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BRAUER GROUPS IN SOME BRAIDED MONOIDAL CATEGORIES

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"Souviens-toi de tout le chemin
que l'Eternel, ton Dieu,
t'a fait faire$\cdots$"
(Deutéronome 8:2).

Se souvenir $\cdots$

pour mieux vivre le présent
et envisager l'avenir.

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ABSTRACT (Résumé)

Dans cette thèse, nous nous sommes intéressés aux groupes de Brauer de certaines catégories monoïdales tressées et à établir des liens entre eux. Ainsi, après avoir énoncé les généralités sur les algèbres de Hopf et sur les notions de catégories dans le chapitre 1 de ce manuscrit, nous avons défini dans le chapitre 2, la notion de groupe de Brauer-Clifford pour la catégorie des (S,H)-dimodules dyslectiques, où H est une algèbre de Hopf commutative et cocommutative et S une algèbre de H-dimodule H-commutative sur un anneau commutatif R. Ce groupe de Brauer est un exemple de groupe de Brauer dans une catégorie monoïdale tressée. Nous avons également montré que ce groupe de Brauer est anti-isomorphe au groupe de Brauer de la catégorie des (S^{op},H)-modules de Hopf-Yetter-Drinfel'd dyslectiques defini par Guédénon et Herman (cf. [32]). Pour une algèbre de Hopf H commutative, cocommutative, projective de type fini comme un R-module, Tilborghs dans [58], a établi un anti-isomorphisme de groupes entre le groupe de Brauer $BD(R,H)$ des H-dimodules et le groupe de Brauer $BD(R,H^*)$ des H^*-dimodules, où H^* est le dual linéaire de H. Nous avons généralisé dans le chapitre 3 ce résultat en établissant un anti-isomorphisme de groupes entre $BD(S,H)$, le groupe de Brauer des algèbres de (S,H)-dimodules dyslectiques et $BD(S^{op},H^*)$, le groupe de Brauer des algèbres (S^{op},H^*)-dimodules dyslectiques, où S est une algèbre de H-dimodule H-commutative et S^{op} est l'algèbre opposée de S. Le chapitre 4 est consacré à la généralisation de la suite de Rosenberg-Zelinsky aux algèbres d'Azumaya des (S,H)-modules de Hopf-Yetter-Drinfel'd dyslectiques dont les termes sont le groupe des automorphismes des S-algèbres H-inner (H-INNER) d'une algèbre $A \in Dys\text{-}_S\mathcal{Q}^H$ et le groupe des classes d'isomorphismes des S-modules inversibles ((S,H)-modules de Hopf-Yetter-Drinfeld dyslectiques inversibles) sous le produit tensoriel $\tilde{\otimes}_S$ noté $Pic(S)$ ($P\mathcal{Q}^H(S,H)$). Lorsque H est une algèbre de Hopf commutative cocommutative, nous avons aussi établi la suite exacte de Rosenberg-Zelinsky des algèbres d'Azumaya des (S,H)-dimodules dyslectiques.

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INTRODUCTION

This thesis is motivated by the importance of Hopf algebras (for a historical overview of Hopf algebras, see [3]) in the theory of "integrals" (Hopf modules [56, Sect. 4.1], Hopf-Galois theory [19], relative Hopf modules [57]), in the theory of Brauer groups (Long dimodules [40]), in the theory of quantum groups (Yetter-Drinfeld modules [62]) and in the theory of invariants (Hopf superalgebras which generalize the supergroup notion), $\cdots$ But we have particularly focused on the study of Brauer groups in some braided monoidal categories.

In 1929, **Richard Brauer** (1901-1977) defined a group that classifies central simple algebras over a given field. Let k be a field. A k-algebra A is said to be central simple if it is simple (it has non-zero ideals) and central (its center is $Z(A) = \{x \in A; x.a = a.x, \forall a \in A\}$ is isomorphic to k). For example, an Azumaya k-algebra is a finite-dimentional central simple algebra. Every field k is central simple over itself. The quaternions ring $\mathbb{H}$, introduced by William Hamilton (1805-1865) is a central simple algebra of rank 4 over $\mathbb{R}$. The field $\mathbb{C}$, of complex numbers, is not central simple because $\mathbb{R}$ is not its center.

The algebra $\mathscr{M}_n(k)$, of $n \times n$ ($n \in \mathbb{N}^*$) matrices with entries in a field k, is an Azumaya k-algebra. If a k-algebra is an Azumaya k-algebra, so is its opposite algebra A^{op} and there is an isomorphism of algebras

$$A \otimes A^{op} \to End(A), \quad (a \otimes b)(x) \mapsto axb,$$

where $\otimes$ denotes the tensor product over R and $End(A)$ the algebra of R-endomorphisms of A. According to R. Brauer in his study of division rings, there is an equivalence relation on the set of Azumaya algebras. Two Azumaya k-algebras A and B are equivalent (or similar), and we write $A \sim B$, if there exist two non-zero integers m and n such that

$$A \otimes \mathscr{M}_n(k) \cong B \otimes \mathscr{M}_m(k)$$

an isomorphism of k-algebras. The equivalence class of an Azumaya k-algebra A is denoted by $[A]$. The set of equivalence classes of Azumaya k-algebras induced by the relation " $\sim$ " with the operation

$$[A].[B] = [A \otimes B]$$

is a group for which the unit element is $[k]$ and the inverse of the class $[A]$ is the class $[A^{op}]$. This group is denoted by $Br(k)$ and it is called the **Brauer group of** k.

The concept of Brauer group of a field was generalized, over time, to various contexts. The Brauer group for a commutative ring was defined by Auslander and Goldman [6] and for a ringed space by Auslander [5]. Grothendieck [26] defined the notion of a bundle of Azumaya algebra for a k-scheme V and constructed the Brauer group of V, denoted today as $BrAz(V)$.

Knus [38] and Childs, Garfinkel and Orzech [20] considered algebras graded by an arbitrary finite abelian group G. They introduced a Brauer group of graded Azumaya algebras, $B_\phi(R,G)$, for G a finite abelian group, R a commutative ring with units group $U(R)$, and $\phi : G\times G \to U(R)$ a fixed bimultiplicative map. When $G = C_2$, the cyclic group of order 2, and ϕ is nontrivial, $B_\phi(R,G)$ is the Brauer-Wall group introduced by Wall [61]. Another generalization of the Brauer group was developed by Fröhlich and Wall [25]. This is the equivariant Brauer group which involves algebras on which a group acts.

In [41], Long introduced $BD(R,G)$, the Brauer group of algebras graded by an arbitrary group G and also acted on by G so that the action preserves the grading. Then, Long computed $BD(R,G)$ for a cyclic group of prime order over an algebraically closed field. In [40], he further extended this to $BD(R,H)$ where H is an arbitrary commutative and cocommutative Hopf algebra over a commutative ring R. This coincides with the former Brauer group in [41] if we let $H = R[G]$, the group algebra of G over R. $BD(R,H)$ is called the Brauer group of H-dimodule algebras or the Brauer-Long group: it consists of equivalence classes of H-Azumaya algebras.

An important step in this sequence of extensions was made by B. Pareigis who, in 1975, defined the Brauer group of a symmetric monoidal category [47]. After, A. Joyal and R. Street introduced, in 1993, the concept of a braided monoidal category [36], F. Van Oystaeyen and Y. Zhang extended, in 1998, Pareigis' definition of the Brauer group [59] to such categories denoted $Br(\mathscr{C})$, where $\mathscr{C}$ is a Braided monoidal category. Before that, the authors with Caenepeel [17] introduced in 1997, $Br({}_H\mathscr{Y}\mathscr{D}^H)$ the Brauer group of the category ${}_H\mathscr{Y}\mathscr{D}^H$ of Yetter-Drinfeld modules over a Hopf k-algebra H. It is denoted $BQ(k,H)$ and it is called the full Brauer group of H. It is this generalization of [59] that we need here, since the category of dyslectic dimodules is braided, not symmetric in general.
Let H be a commutative cocommutative Hopf algebra over a commutative ring R. The category $\mathscr{D}^H$ of H-dimodules is a braided monoidal category, so one can consider (commutative) algebras in it. These algebras are termed (quantum commutative) H-dimodule algebras. Let S be such a quantum commutative H-dimodule algebra. An (S,H)-dimodule is an H-dimodule with a left S-action satisfying some compatibility conditions. We can consider the category ${}_S\mathscr{D}^H$ of (S,H)-dimodules. Using the braiding on $\mathscr{D}^H$, the left S-action on an (S,H)-dimodule M can be used to define a right S-action on it, making M an S-bimodule. Then the tensor product $M\otimes_S N$ of two (S,H)-dimodules M and N is again an (S,H)-dimodule, see [33, Lemma 3.1]. Thus we obtain a monoidal category $({}_S\mathscr{D}^H, \otimes_S, S)$. This category is not braided; the reason is that the braiding map $\gamma_{M,N} : M\otimes N \to N\otimes M$ does not induce a well-defined map $\gamma_{M,N} : M\otimes_S N \to N\otimes_S M$. However, if M and N are dyslectic, then there is no problem. Actually this is how the definition of dyslexia modules was designed by Pareigis in [50]. The full subcategory $Dys\text{-}{}_S\mathscr{D}^H$ of ${}_S\mathscr{D}^H$ consisting of dyslectic (S,H)-dimodules is a braided monoidal category. There are some additional properties making the category closed, see the Lemmas in [33, Sect. 4]. With these properties, according to [59], we can define the Brauer group of the category of dyslectic (S,H)-dimodules $Br(Dys\text{-}{}_S\mathscr{D}^H)$ denoted by $BD(S,H)$. For more recent versions of the Brauer group, see [32], [34],[27], [55], [35], [28],[29], [30], [53], [11],$\cdots$ The list is not exhaustive.

The content of this thesis is divided into four chapters. In Chapter 1 we give the generalities on Hopf algebras, monoidal categories, symmetric monoidal categories, braided monoidal categories and their functors which we use to establish some of our results. The Chapter 2 is the subject of an article [33] entitled "*A Brauer-Clifford-Long group for the category of dyslectic (S,H)-dimodule algebras.*" In this one, we considered the monoidal category ${}_S\mathscr{D}^H$ of (S,H)-dimodules, where S is a H-dimodule algebra H-commutative according to the braiding of dimodules. As this category being neither symmetric nor braided, the notion of dyslexia ([50]) is used to make it monoidal braided. This gives us the theorem:

Theorem 0.0.1. *[33, Theorem 4.4]*
The category $(\mathscr{D}ys\text{-}{}_S\mathscr{D}^H, \tilde{\otimes}_S, S, \gamma)$ of dyslectic (S,H)-dimodules is a braided monoidal category.

We define the Azumaya algebras in the category $(\mathscr{D}ys\text{-}{}_S\mathscr{D}^H, \tilde{\otimes}_S, S, \gamma)$ and obtain the main result of this chapter:

Theorem 0.0.2. *[33, Theorem 6.5]*
Let H be a commutative and cocommutative Hopf algebra and S an H-commutative H-dimodule algebra. Then $BD(S,H)$ is a group. If $[A]$ and $[B]$ denote the equivalence classes of dyslectic (S,H)-dimodules Azumaya algebras A and B, then in $BD(S,H)$, we will have $[A].[B] = [A\#_S B]$. The identity of $BD(S,H)$ is the equivalence class $[S]$ consisting of all trivial dyslectic (S,H)-dimodules Azumaya algebras, and $[A]^{-1} = [\bar{A}]$, for all $[A] \in BD(S,H)$.

This group generalizes the Brauer group $BD(R,H)$ defined by Long in [40]. In 2018, Guédénon and Herman in [32] introduced the category dyslectic Hopf Yetter-Drinfel'd (S,H)-modules denoted $Dys\text{-}{}_S\mathscr{Q}^H$ and its Brauer group $BQ(S,H)$ called Brauer-Clifford group. In this chapter, we establish a link between the groups $BD(S,H)$ and $BQ(T,H)$ as follows:

Theorem 0.0.3. *[33, Theorem 8.7]*
Let H be a Hopf algebra and S be an H-commutative H-dimodule algebra. There is an anti-isomorphism of groups

$$\chi : BD(S,H) \to BQ(S^{op},H) \text{ given by } \chi([A]) = \left[\left[A^{op}\right]\right],$$

where $[[A^{op}]]$ represents the class of A^{op} in $BQ(S^{op},H)$.

In Chapter 3 (which is an article [46], accepted for publication in *Beiträge zur Algebra und Geometrie*), we consider a commutative cocommutative Hopf algebra H which is finitely generated projective over a commutative ring R. If so, the dual of H , H^* is also a commutative and cocommutative Hopf algebra. We show that if an R-module S is an H-commutative H-dimodule algebra then, its natural opposite algebra S^{op} is an H^*-commutative H^*-dimodule. This leads us to deduce the braided monoidal category $(Dys\text{-}{}_{S^{op}}\mathscr{D}^{H^*}, \tilde{\otimes}_{S^{op}}, S^{op}, \gamma^*)$ of dyslectic (S^{op},H^*)-dimodule algebras and its Brauer group $BD(S^{op},H^*)$. In this chapter, we have generalized Tilborghs' result [58] by the theorem:

Theorem 0.0.4. *The functor*

$$(\mathscr{F}, \varphi_0, \varphi_2) : Dys\text{-}{}_S\mathscr{D}^{H\text{-}rev} \to Dys\text{-}{}_{S^{op}}\mathscr{D}^{H^*}$$

is an isomorphism of braided monoidal categories. Consequently

$$BD(S,H)^{op} \cong BD(S^{op},H^*),$$

as isomorphism of groups. This means that $BD(S,H)$ *and* $BD(S^{op},H^*)$ *are anti-isomorphic Brauer-Clifford-Long groups.*

The Chapter 4 of this thesis is devoted to the Rosenberg-Zelinsky exact sequence which links the Picard group to the classes of automorphisms of Azumaya algebras for the categories $Dys\text{-}_S\mathcal{Q}^H$ and $Dys\text{-}_S\mathcal{D}^H$ separately. These sequences are defined as follows.

Theorem 0.0.5. *Let* A *be a dyslectic Hopf Yetter-Drinfeld* (S,H)*-module Azumaya algebra. Then there are exact sequences of groups*

$$1 \to H\text{-}Inn(A) \to H\text{-}\mathrm{Aut}_S(A) \xrightarrow{\Psi} Pic(S) \tag{0.0.1}$$

and

$$1 \to H\text{-}INN(A) \to H\text{-}\mathrm{Aut}_S(A) \xrightarrow{\Phi} P\mathcal{Q}^H(S,H). \tag{0.0.2}$$

The homomorphisms Ψ *and* Φ *are respectively defined by* $\Psi(\alpha) = [G_H({}_1A_\alpha)]$ *and* $\Phi(\alpha) = \{G_H({}_1A_\alpha)\}$*, for every* $\alpha \in H\text{-}\mathrm{Aut}_S(A)$*, where* G_H *is the inverse of the equivalence functor* $F_H : N \to A \otimes_S N$*.*

Theorem 0.0.6. *Let* A *be a dyslectic* (S,H)*-dimodule Azumaya algebra. Then there are exact sequences of groups*

$$1 \to H\text{-}Inn(A) \to H\text{-}\mathrm{Aut}_S(A) \xrightarrow{\Psi'} Pic(S) \tag{0.0.3}$$

and

$$1 \to H\text{-}INN(A) \to H\text{-}\mathrm{Aut}_S(A) \xrightarrow{\Phi'} P\mathcal{D}^H(S,H). \tag{0.0.4}$$

The homomorphisms Ψ' *and* Φ' *are respectively defined by* $\Psi'(\alpha) = [[G_H({}_1A_\alpha)]]$ *and* $\Phi'(\alpha) = \{\{G_H({}_1A_\alpha)\}\}$*, for every* $\alpha \in H\text{-}\mathrm{Aut}_S(A)$*, where* G_H *is the inverse of the equivalence functor* $F_H : N \to A \otimes_S N$*.*

Chapter 1

GENERALITIES ON HOPF ALGEBRAS

In all this text, R is a commutative ring, the unspecified tensor product is on R and all maps are R-linear (unless specified).

1.1 Coalgebras

Definition 1.1.1. *An R-algebra $A = (A, m_A, \mu_A)$ is a triplet where A is an R-module and the maps $m_A : A \otimes A \to A; a \otimes a' \mapsto a.a' = aa'$ and $\mu_A : R \to A; \lambda \mapsto \lambda 1_A$ are R-linear maps such that the following diagrams are commutative:*

- *the associativity*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "id_A \otimes m_A"] \arrow[d, "m_A \otimes id_A"'] & A \otimes A \arrow[d, "m_A"] \\ A \otimes A \arrow[r, "m_A"'] & A \end{tikzcd}$$

- *the unity*

$$\begin{tikzcd} & A \otimes A \arrow[dd, "m_A"] & \\ R \otimes A \arrow[ur, "\mu_A \otimes id_A"] \arrow[dr, "id_A"'] & & A \otimes R \arrow[ul, "id_A \otimes \mu_A"'] \arrow[dl, "id_A"] \\ & A & \end{tikzcd}$$

The map m_A is called the product or multiplication, the map μ_A is the unity map and $\mu_A(1_R)$ is the unit element of A.

Let (A, m_A, μ_A) be an R-algebra. We set $A^{op} = (A, m_A^{op}, \mu_A)$, where $m_A^{op} = \tau \circ m_A$ and $\tau : A \otimes A' \to A' \otimes A$ denotes the flip map $\tau(a \otimes a') = a' \otimes a$. Then A^{op} is an algebra, called the opposite algebra of A. An algebra (A, m_A, μ_A) is said to be commutative if $m_A = m_A^{op}$.

Definition 1.1.2. *A morphism of R-algebras is a map $f : A \to A'$ such that the following diagrams are commutative:*

$$\begin{tikzcd}
A \otimes A \arrow[r, "m_A"] \arrow[d, "f \otimes f"'] & A \arrow[d, "f"] \\
A' \otimes A' \arrow[r, "m_{A'}"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mu_A"'] & A'; \arrow[dl, "\mu_{A'}"'] \\
R &
\end{tikzcd}$$

A coalgebra is the dual notion of algebra. It is defined by reversing the arrows in the definition of algebra.

Definition 1.1.3. *A co-algebra (or coalgebra) C is a triplet $(C, \Delta_C, \varepsilon_C)$; where C is an R-module, $\Delta_C : C \longrightarrow C \otimes C$ and $\varepsilon_C : C \longrightarrow R$ are R-linear maps such that the following diagrams are commutative:*

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "id_C \otimes \Delta_C"'] \\
C \otimes C \arrow[u, "\Delta_C \otimes id_C"] & C \arrow[l, "\Delta_C"] \arrow[u, "\Delta_C"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& C \otimes C \arrow[dl, "\varepsilon_C \otimes id_C"'] \arrow[dr, "id_C \otimes \varepsilon_C"] & \\
R \otimes C & & C \otimes R; \\
& C \arrow[ul, "id_C"] \arrow[ur, "id_C"'] \arrow[uu, "\Delta_C"] &
\end{tikzcd}$$

which translates to:

- *the co-associativity:*
$$(\Delta_C \otimes id_C) \circ \Delta_C = (id_C \otimes \Delta_C) \circ \Delta_C,$$
- *the co-unity:*
$$(id_C \otimes \varepsilon_C) \circ \Delta_C = (\varepsilon_C \otimes id_C) \circ \Delta_C.$$

The map Δ_C is called the co-multiplication or the co-product of C and the map ε_C is called the co-unit of C.

***Sweedler notations [56]*:**
Let $C = (C, \Delta_C, \varepsilon_C)$ be a coalgebra. An element of $C \otimes C$ is of the form $\sum_{i=1}^{n} c_i \otimes d_i$. For uniformity of writing and by convention, we use the Sweedler-Heyneman notation: let $c \in C$, we denote

$$\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)} = \sum c_1 \otimes c_2 = c_{(1)} \otimes c_{(2)} = c_1 \otimes c_2.$$

Sweedler-Heyneman (or Sweedler) notations are very useful for doing calculations in coalgebras. In the rest of this work, we will use the notation $\Delta_C(c) = c_1 \otimes c_2$. With this notation, the axiom of co-associativity is translated as:

$$\Delta_C(c_1) \otimes c_2 = c_1 \otimes \Delta_C(c_2);$$

that is,

$$c_{11} \otimes c_{12} \otimes c_2 = c_1 \otimes c_{21} \otimes c_{22} = c_1 \otimes c_2 \otimes c_3, \quad \forall c \in C.$$

The axiom of co-unity translates into:

$$\varepsilon_C(c_1)c_2 = c = c_1\varepsilon_C(c_2), \quad \forall c \in C.$$

Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra. We set $C^{cop} = (C, \Delta_C^{op}, \varepsilon_C)$, where $\Delta_C^{op} = \tau \circ \Delta_C$. C^{cop} is a coalgebra called the co-oppisite coalgebra of C. A coalgebra C is said to be cocommutative if $\Delta_C = \Delta_C^{op}$.

Definition 1.1.4. *Let $C = (C, \Delta_C, \varepsilon_C)$ and $D = (D, \Delta_D, \varepsilon_D)$ be two coalgebras. We define two R-linear maps $\Delta_{C\otimes D} : C \otimes D \longrightarrow C \otimes D \otimes C \otimes D$ and $\varepsilon_{C\otimes D} : C \otimes D \longrightarrow R$ given by:*

$$\Delta_{C\otimes D} = (id_C \otimes \tau_{C\otimes D} \otimes id_D) \circ (\Delta_C \otimes \Delta_D) \quad and \quad \varepsilon_{C\otimes D} = \varepsilon_C \otimes \varepsilon_D.$$

In other words we have:

$$\Delta_{C\otimes D}(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2 \quad and \quad \varepsilon_{C\otimes D}(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d),$$

where $\Delta_C(c) = c_1 \otimes c_2$, $\Delta_D(d) = d_1 \otimes d_2$.

Proposition 1.1.5. *The triplet $(C \otimes D, \Delta_{C\otimes D}, \varepsilon_{C\otimes D})$ defined above is a coalgebra called the tensor product of coalgebras C and D.*

Definition 1.1.6. *Let C and D be two coalgebras. An R-linear map $f : C \longrightarrow D$ is a morphism of coalgebras if the following diagrams*

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "f"'] & C \otimes C \arrow[d, "f\otimes f"] \\ D \arrow[r, "\Delta_D"'] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\varepsilon_C"'] & D; \arrow[dl, "\varepsilon_D"] \\ R & \end{tikzcd}$$

are commutative. That is: $(f \otimes f) \circ \Delta_C = \Delta_D \circ f$ and $\varepsilon_D \circ f = \varepsilon_C$, in other words

$$f(c)_1 \otimes f(c)_2 = f(c_1) \otimes f(c_2) \quad and \quad \varepsilon_D(f(c)) = \varepsilon_C(c) \quad \forall c \in C.$$

Remark 1.1.7. *1. Let (A, m_A, μ_A) be a finite dimensional algebra, then its dual A^* is a coalgebra with comultiplication*

$$\Delta_{A^*} : A^* \xrightarrow{m_A^*} (A \otimes A)^* \xrightarrow{\cong} A^* \otimes A^*$$

and counity $\varepsilon_{A^*} = \mu_A^*$.

2. *Let $(C,\Delta_C,\varepsilon_C)$ be a coalgebra not necessarily of finite dimension. Its dual C^* is an algebra with product*

$$m_{C^*}: C^*\otimes C^* \xrightarrow{\cong} (C\otimes C)^* \xrightarrow{\Delta_C^*} C^*$$

and the unity is $\mu_{C^*}=\varepsilon_C^*$.

Proposition 1.1.8. *[22, Proposition 4.1.1] Let (B,m_B,μ_B) be an algebra and $B=(B,\Delta_B,\varepsilon_B)$ be a coalgebra. The following assertions are equivalent:*

1. *The maps m_B and μ_B are morphisms of coalgebras.*

2. *The maps Δ_B and ε_B are morphisms of algebras.*

Definition 1.1.9. *A bialgebra B is a quintuplet $(B,m_B,\mu_B,\Delta_B,\varepsilon_B)$ such that (B,m_B,μ_B) is an algebra, $B=(B,\Delta_B,\varepsilon_B)$ is a coalgebra and the maps Δ_B and ε_B are morphisms of algebras (or equiventely the maps m_B and μ_B are morphisms of coalgebras).*
A bialgebra morphism $f:B\to B'$ is a linear map which is both a morphism of algebras and cogrebras.

Definition 1.1.10. *Let B be a bialgebra. We say that a nonzero element x of B is a* ***group-like element*** *if:*

$$\Delta_B(x)=x \quad and \quad \varepsilon_B(x)=1_R.$$

*We say that an element x of B is a (g,h)-**primitive element** if:*

$$\Delta_B(x)=g\otimes x+x\otimes h \quad and \quad \varepsilon_B(x)=0$$

where g and h are group-like elements of B. An element x of a bialgebra B is said to be a ***primitive element*** *if:*

$$\Delta_B(x)=x\otimes 1_B+1_B\otimes x \quad and \quad \varepsilon_B(x)=0.$$

Remark 1.1.11. 1. *If $(B,m_B,\mu_B,\Delta_B,\varepsilon_B)$ is a bialgebra, then $B^{op}=(B,m_B^{op},\mu_B,\Delta_B,\varepsilon_B)$, $B^{cop}=(B,m_B,\mu_B,\Delta_B^{op},\varepsilon_B)$ and $B^{op,cop}=(B,m_B^{op},\mu_B,\Delta_B^{op},\varepsilon_B)$ are bialgebras.*

2. *If B is a finite dimentional bialgebra, then its dual $B^*=(B^*,m_{B^*},\mu_{B^*},\Delta_{B^*},\varepsilon_{B^*})$ is a bialgebra (cf. [22, Proposition 4.1.6])*

1.2 Hopf algebras

Let (A,m_A,μ_A) be an algebra and $(C,\Delta_C,\varepsilon_C)$ a coalgebra. Let $Hom_R(C,A)$ be the set of R-linear maps from C to A.

Definition 1.2.1. *Let $f,g \in Hom_R(C,A)$. The convolution is defined by :*

$$f * g = m_A \circ (f \otimes g) \circ \Delta_C;$$

which translates into the commutativity of the diagram:

$$\begin{tikzcd}
C \otimes C \arrow[r, "f\otimes g"] & A \otimes A \arrow[d, "m_A"] \\
C \arrow[u, "\Delta_C"] \arrow[r, "f*g"'] & A
\end{tikzcd}$$

In other words, with the Sweedler notation, for all $c \in C$ we have:

$$(f * g)(c) = f(c_1)g(c_2).$$

This product is called the convolution product.

Proposition 1.2.2. *Let (A, m_A, μ_A) be an algebra and $(C, \Delta_C, \varepsilon_C)$ be a coalgebra. $(Hom_R(C,A); *)$ is a unitary algebra of unit element $\mu_A \circ \varepsilon_C$. In particular, the dual $C^* = Hom_R(C,R)$ of C is a unitary algebra of unit element $\mu_R \circ \varepsilon_C = \varepsilon_C$.*

Definition 1.2.3. *Let $H = (H, m_H, \mu_H, \Delta_H, \varepsilon_H)$ be a bialgebra. Let $End_R(H)$ be the set of endomorphisms of H. Its unit element is $\mu_H \circ \varepsilon_H$. The unique inverse of id_H (if it exists) in $End_R(H)$, called the **antipode** of H is the map $S_H : H \to H$. So $S_H \in End_R(H)$ and*

$$S_H * id_H = \mu_H \circ \varepsilon_H = id_H * S_H.$$

Definition 1.2.4. *A Hopf algebra is a sextuplet $H = (H, m_H, \mu_H, \Delta_H, \varepsilon_H, S_H)$, where $(H, m_H, \mu_H, \Delta_H, \varepsilon_H)$ is a bialgebra and $S_H : H \to H$ is a linear map, called the **antipode** (or the **coinverse**) of H, such that the following diagram is commutative:*

$$\begin{tikzcd}
 & H \otimes H \arrow[r, "S_H\otimes id_H"] & H \otimes H \arrow[dr, "m_H"] & \\
H \arrow[ur, "\Delta_H"] \arrow[r, "\varepsilon_H"] \arrow[dr, "\Delta_H"'] & R \arrow[rr, "\mu_H"] & & H \\
 & H \otimes H \arrow[r, "id_H\otimes S_H"'] & H \otimes H \arrow[ur, "m_H"'] &
\end{tikzcd}$$

This means that,

$$m_H \circ (S_H \otimes id_H) \circ \Delta_H = \mu_H \circ \varepsilon_H = m_H \circ (id_H \otimes S_H) \circ \Delta_H.$$

With Sweedler's notations, it is equivalent to:

$$S_H(h_1)h_2 = \varepsilon_H(h)1_H = h_1 S_H(h_2) \quad \forall h \in H. \tag{1.2.1}$$

The formula (1.2.1) is called the antipode formula.

Theorem 1.2.5. ***(The properties of the antipode)***
Let H be a Hopf algebra with antipode S_H:

i) $S_H : H \to H^{op}$ *is an algebras homomorphism:*

$$S_H(gh) = S_H(h)S_H(g), \qquad S_H(1_H) = 1_H \quad \forall h, g \in H. \tag{1.2.2}$$

ii) $S_H : H \to H^{cop}$ *is a coalgebras homomorphism:*

$$\Delta_H(S_H(h)) = S_H(h_2) \otimes S_H(h_1), \qquad \varepsilon_H(S_H(h)) = \varepsilon_H(h) \quad \forall h \in H. \tag{1.2.3}$$

iii) If H is commutative or cocommutative, then $S_H^2 = id_H$, that is, $S_H^{-1} = S_H$.

Corollary 1.2.6. *Let $(H, m_H, \mu_H, \Delta_H, \varepsilon_H, S_H)$ be a Hopf algebra with a bijective antipode. Then*

$$H^{op} = (H, m_H^{op}, \mu_H, \Delta_H, \varepsilon_H, S_H^{-1})$$

is a Hopf algebra.

Proposition 1.2.7. *Let H and H' be Hopf algebras with antipodes S_H and $S_{H'}$ respectively. The tensor product $H \otimes H'$ is a Hopf algebra with antipode $S_{H\otimes H'} = S_H \otimes S_{H'}$.*

Lemma 1.2.8. *Let $H = (H, m_H, \mu_H, \Delta_H, \varepsilon_H, S_H)$ be a Hopf algebra with bijective antipode S_H. Then,*

$$S_H^{-1}(h_2)h_1 = \varepsilon_H(h)1_H = h_2 S_H^{-1}(h_1), \quad \forall h \in H.$$

Definition 1.2.9. *Let H and H' be two Hopf algebras with antipodes S_H and $S_{H'}$ respectively. An R-linear map $f : H \longrightarrow H'$ is a morphism of Hopf algebras if f is a morphism of algebras, a morphism of coalgebras such that:*

$$S_{H'} \circ f = f \circ S_H.$$

Theorem 1.2.10. ***(Dual of a Hopf algebra)***
Let $H = (H, m_H, \mu_H, \Delta_H, \varepsilon_H, S_H)$ be a finite dimensional Hopf algebra. Then its dual

$$H^* = (H^*, m_{H^*}, \mu_{H^*}, \Delta_{H^*}, \varepsilon_{H^*}, S_{H^*})$$

is a Hopf algebra with antipode $S_{H^*} = S_H^*$ given by:

$$S_{H^*}(f) = f \circ S_H, \quad \forall f \in H^*.$$

Definition 1.2.11. ***(H-invariant elements)***
Let H be a Hopf algebra and let M be a left H-module. An element m of M is called H-invariant if:

$$h.m = \varepsilon_H(h)m, \quad \forall h \in H.$$

The set of H-invariant elements of M is denoted by M^H. Therefore

$$M^H = \{m \in M, h.m = \varepsilon_H(h)m, \quad \forall h \in H\}.$$

An R-module M is a trivial H-module if:

$$h.m = \varepsilon_H(h)m, \quad \forall m \in M, h \in H.$$

Therefore M^H is a trivial left H-module. Moreover M^H is a sub-R-module of M.

1.3 Some elementary examples of Hopf algebra

Let $\Bbbk$ be a field.

Example 1.3.1. ***The algebra*** $\Bbbk G$ *[1, Example 2.7]*
Let G be a group. We consider the algebra $\Bbbk G$ of the group G on $\Bbbk$, which is a $\Bbbk$-vector space with basis G. Then $\Bbbk G = (\Bbbk G, m, \mu, \Delta, \varepsilon, S)$ is a Hopf algebra where:

- *the product*
$$m : \Bbbk G \otimes \Bbbk G \to \Bbbk G; \qquad (x, y) \mapsto xy,$$
- *the unit*
$$\mu : \Bbbk \to \Bbbk G; \qquad \lambda \mapsto \lambda 1_G,$$
- *the coproduct*
$$\Delta : \Bbbk G \to \Bbbk G \otimes \Bbbk G; \qquad x \mapsto x \otimes x,$$
- *the counit*
$$\varepsilon : \Bbbk G \to G; \qquad x \mapsto 1$$
- *and the antipode*
$$S : \Bbbk G \to \Bbbk G, \qquad x \mapsto x^{-1}.$$

Since $\Delta(x) = x \otimes x$, then $\Bbbk G$ is a cocommutative Hopf algebra. It is commutative if and only if G is an abelian group. The set of group-like elements of $\Bbbk G$ is G.

Example 1.3.2. ***The algebra*** $\Bbbk^G$
Let G be again a multiplicative finite group. Let us consider the $\Bbbk$-algebra $\Bbbk^G$ of the maps $G \to \Bbbk$. As a $\Bbbk$-vector space, a basis of $\Bbbk^G$ is formed by δ_g ($g \in G$), with the Kronecker symbol

$$\delta_g(g') = \begin{cases} 1 & \text{if } g' = g \\ 0 & \text{if } g' \neq g. \end{cases}$$

We define the product m by
$$m(\delta_g \otimes \delta_{g'}) = \delta_g \delta_{g'}.$$
Thus the product m is commutative and $m(\delta_g \otimes \delta_g) = \delta_g$ for $g \in G$. the unit
$$\mu : \Bbbk \to \Bbbk^G \text{ sends 1 to } \sum_{g \in G} \delta_g.$$

The coproduct $\Delta : \Bbbk^G \to \Bbbk^G \otimes \Bbbk^G$ and the counit $\varepsilon : \Bbbk^G \to \Bbbk$ are defined by:

$$\Delta(\delta_g) = \sum_{g'g''=g} \delta_{g'} \otimes \delta_{g''} \qquad and \qquad \varepsilon(\delta_g) = \delta_g(1_G).$$

This coproduct is cocommutative if and only if the group G is commutative. The antipode $S : \Bbbk^G \to \Bbbk^G$ is given by
$$S(\delta_g) = \delta_{g^{-1}}.$$
With these maps, $\Bbbk^G$ is a Hopf algebra.

Remark 1.3.3.
- *We can identify $\Bbbk^G \otimes \Bbbk^G$ with $\Bbbk^{G\times G}$ when we set:*

$$\delta_g \otimes \delta_{g'} = \delta_{(g,g')}.$$

- *The Hopf algebras $\Bbbk G$ and $\Bbbk^G$ are dual of each other: a non-degenerate mating is given by*

$$\Bbbk G \times \Bbbk^G \to \Bbbk; \qquad (g_1, \delta_{g_2}) \mapsto \delta_{g_2}(g_1).$$

The basis G of $\Bbbk G$ is dual to the basis $(\delta_g)_{g\in G}$ of $\Bbbk^G$.

Example 1.3.4. ***The algebra*** $\mathfrak{U}(\mathfrak{g})$ *[1, Example 2.8]*
Let $\mathfrak{g}$ be a Lie algebra with a law classically denoted by the bracket $[.,.]$. We denote by $\mathfrak{U}(\mathfrak{g})$ its universal enveloping algebra, which is the quotient $\mathfrak{T}(\mathfrak{g})/\mathfrak{I}$ of the tensorial algebra $\mathfrak{T}(\mathfrak{g})$ of $\mathfrak{g}$ by the two-sided ideal $\mathfrak{I}$ generated by the elements of the form $XY - YX - [X,Y]$. We define a coproduct Δ, a counit ε and an antipode S_H over $\mathfrak{U}(\mathfrak{g})$ by the relations:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \varepsilon(x) = 0 \qquad \textit{and} \qquad S(x) = -x$$

for $x \in \mathfrak{g}$. Thus $\mathfrak{U}(\mathfrak{g})$ is a cocommutative Hopf algebra. The set of primitive elements of $\mathfrak{U}(\mathfrak{g})$ is nothing else $\mathfrak{g}$.

Example 1.3.5. ***The algebra*** $\Re(G)$
Let G be a compact topological group over $\mathbb{C}$. We denote by $\Re(G)$ the set of continuous maps $f : G \to \mathbb{C}$ such that, with t describing G, the translates $f_t : x \mapsto f(tx)$ generate a finite-dimensional vector space. We define a coproduct Δ, a counit ε and an antipode S_H over $\Re(G)$ by the relations:

$$\Delta f(x,y) = f(xy) \qquad \varepsilon(f) = f(1) \qquad \textit{and} \qquad Sf(x) = f(x^{-1}),$$

for all $x, y \in G$. Therefore $\Re(G)$ is a commutative Hopf algebra.

Example 1.3.6. ***The algebra*** $\Bbbk(G)$ *[43, Example 2.1]*
Let G be a finite group with identity e. Let us denote by $\Bbbk(G)$ the algebra of functions $f : G \to \Bbbk$ with the product

$$(fg)(x) = f(x)g(x); \quad \forall x \in G \textit{ and } f, g \in \Bbbk(G).$$

$\Bbbk(G)$ is a Hopf algebra called the group function Hopf algebra, its coproduct is

$$\Delta : \Bbbk(G) \to \Bbbk(G) \otimes \Bbbk(G) \cong \Bbbk(G \times G); \qquad (\Delta f)(x,y) = f(xy)$$

its counit is

$$\varepsilon : \Bbbk(G) \to \Bbbk; \qquad \varepsilon(f) = f(e)$$

and its antipode is

$$S : \Bbbk(G) \to \Bbbk(G); \qquad (Sf)(x) = f(x^{-1}).$$

Example 1.3.7. ***The algebra*** $\Bbbk[SL_2]$ *[43, Example 2.2]*
The Hopf algebra $\Bbbk[SL_2]$ is $\Bbbk[a,b,c,d]$ modulo the relation

$$det(M) = 1, \textit{ where we set } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \textit{ and } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The coproduct, counit and antipode are

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = b \otimes d + a \otimes b, \quad \Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = d \otimes d + c \otimes b$$

$$\varepsilon(a) = \varepsilon(d) = 1, \qquad \varepsilon(b) = \varepsilon(c) = 0$$

$$S(a) = d, \qquad S(b) = c, \qquad S(c) = b, \qquad S(d) = a$$

$$\Delta(M) = M \otimes M, \qquad \varepsilon(M) = I_2 \qquad and \qquad S(M) = M^{-1};$$

where matrix multiplication should be understood in this definition of A.

1.4 Modules

Definition 1.4.1. *Let A be an R-algebra. An R-module M is a left A-module if there is an R-linear map:*

$$\begin{aligned} \lambda_M : \quad & A \otimes M \longrightarrow M \\ & a \otimes m \longmapsto a.m = am, \quad \forall a \in A, m \in M \end{aligned}$$

such that the following diagrams are commutative

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "id_A \otimes \lambda_M"] \arrow[d, "m_A \otimes id_M"'] & A \otimes M \arrow[d, "\lambda_M"] \\ A \otimes M \arrow[r, "\lambda_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \otimes M \arrow[rr, "\mu_A \otimes id_M"] \arrow[dr, "\approx"'] & & A \otimes M \arrow[dl, "\lambda_M"] \\ & M & \end{tikzcd}$$

that is, for the rectangle: $(ab)m = a(bm)$ and for the triangle: $1_A m = m$, for all $a, b \in A, m \in M$. The map λ_M is called the left A-action on M. It is also denoted in this manuscrit by $\rightharpoonup$.

We denote by ${}_A\mathscr{M}$ (or $\mathscr{M}_A$) the category of left (or right) A-modules. Its objects are the left (or right) A-modules and homomorphisms are the left (or right) A-modules homomorphisms.

Definition 1.4.2. *Let A un an algebra and M and N two left A-modules. An A-module hohomorphism $f : M \to N$ is an R-linear map which makes commutative the following diagram:*

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ A \otimes M \arrow[u, "\lambda_M"] \arrow[r, "id_A \otimes f"'] & A \otimes N, \arrow[u, "\lambda_N"'] \end{tikzcd}$$

This means that $f \circ \lambda_M = \lambda_N \circ (id_A \otimes f)$. In other words, for $a \in A$ and $m \in M$, a map f is said to be an A-modules homomorphism if

$$f(am) = af(m).$$

Definition 1.4.3. ***(Bimodule)***
Let A and B be two R-algebras. An R-module M is said to be an $(A$-$B)$-bimodule and denoted by ${}_AM_B$, if M is a left A-module and a right B-module, and the two multiplications are related by an associative law (or the compatibility):

$$(am)b = a(mb), \quad \forall a \in A, b \in B, m \in M.$$

Example 1.4.4.

1. *Every algebra A is an $(A$-$A)$-bimodule, the compatibility is just the associativity of multiplication in A. More generally, if $B \subseteq A$ is a sub-algebra, then A is an $(A$-$B)$-bimodule.*
2. *If A is commutative, then every left (or right) A-module is an $(A$-$A)$-bimodule.*

Definition 1.4.5. ***(Finitely generated R-module)***
*A **finitely generated** R-module M is an R-module that has a finite generating set. A finitely generated module over a ring R may also be called a finite R-module, finite over R, or an R-module of finite type. That is, there exist*

$$m_1, m_2, m_3, \cdots, m_l \in M \text{ such that } M = \sum_{i=1}^{l} Rm_i.$$

Definition 1.4.6. ***(Free R-module)***
*A left R-module M is said to be **left free R-module** is isomorphic to a direct sum of copies of R, that is, there is a (possibly infinite) index set B with $M = \bigoplus_{b\in B} R_b$, where $R_b = \langle b \rangle \cong R$ for all $b \in B$. B is called the basis of M.*

By the definition of direct sum, each $m \in M$ has a unique expression of the form

$$m = \sum_{b \in B} r_b b$$

where $r_b \in R$ and almost all $r_b = 0$. It follows that $M = \langle B \rangle$.

*A free $\mathbb{Z}$-module is called a **free abelian group**. Every ring R, when considered as a left module over itself, is itself a free R-module.*

Definition 1.4.7. ***(Projective R-module)***
*Let M and N be two left R-modules. A left R-module P is **projective** if and only if for every surjective R-module homomorphism $f : N \to M$ and every R-module homomorphism $g : P \to M$, there exists an R-module homomorphism $h : P \to N$ such that $g = f \circ h$, that is, there exists a map $h : P \to N$ making the following diagram commutative:*

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dotted, "h"'] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Theorem 1.4.8. *([52, Theorem 3.5])*

i) A left R-module P is projective if and only if P is a direct summand of a free left R-module.

ii) A finitely generated left R-module P is projective if and only if P is a direct summand of R^n for some n.

Definition 1.4.9. ***(Faithfully projective R-module)***
An R-module M is said to be ***faithful*** *if for all distinct elements a,b of R, there exists $x \in M$ such that $ax \neq bx$. In other words, the multiplications by a and by b define two different endomorphisms of M.*

A left R-module is ***faithfully projective*** *if it is finitely generated projective and faithfull as a left R-module.*

1.5 Comodules

The notions of comodules and comodule morphisms are dual notions of modules and module morphisms.

Definition 1.5.1. *Let M be a left (right) A-module.We denote by ${}_A\mathcal{M}$ ($\mathcal{M}_A$) the category of left (right) A-modules. Objects are left (right) A-modules and morphisms are morphisms of left (right) A-modules.*

Definition 1.5.2. *Let $C = (C, \Delta_C, \varepsilon_C)$ be a coalgebra. A left R-module M is a right C-comodule if there exists a R-linear map $\varphi_M : M \longrightarrow M \otimes C$, which makes the following diagrams commutative:*

$$
\begin{tikzcd}
M \arrow[r, "\varphi_M"] \arrow[d, "\varphi_M"'] & M \otimes C \arrow[d, "id_M \otimes \Delta_C"] \\
M \otimes C \arrow[r, "\varphi_M \otimes id_C"'] & M \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\varphi_M"] \arrow[dr, "\cong"'] & M \otimes C \arrow[d, "id_M \otimes \varepsilon_C"] \\
 & M \otimes R
\end{tikzcd}
$$

The map φ_M is called the C-coaction or the coaction of C on M.

Let $M = (M, \varphi_M)$ be a right C-comodule. For $m \in M$, we have the Sweedler's notations:

$$\varphi_M(m) = \Sigma m_{(0)} \otimes m_{(1)} = \Sigma m_0 \otimes m_1 = m_0 \otimes m_1.$$

In the remainder of our work, we will use the notation

$$\varphi_M(m) = m_0 \otimes m_1, \quad \forall m \in M.$$

With Sweedler's notation, the commutativity of the previous diagrams is equivalent to

$$m_0 \otimes m_{11} \otimes m_{12} = m_{00} \otimes m_{01} \otimes m_1 = m_0 \otimes m_1 \otimes m_2$$

and

$$m_0 \varepsilon_C(m_1) = m \qquad \forall m \in M.$$

Definition 1.5.3. *Let C be a coalgebra and M a right C-comodule. A sub-R-module N of M is a right sub-C-comodule of M if*

$$\varphi_M(N) \subseteq N \otimes C.$$

Definition 1.5.4. *Let C be a coalgebra and let M and N be two right C-comodules. A R-linear map $f : M \longrightarrow N$ is a morphism of C-comodules or a C-colinear map if the following diagram is commutative.*

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\varphi_M"'] & N \arrow[d, "\varphi_N"] \\
M \otimes C \arrow[r, "f \otimes id_C"'] & N \otimes C,
\end{tikzcd}$$

that is: $\varphi_N \circ f = (f \otimes id_C) \circ \varphi_M$. Then, for all $m \in M$,

$$(\varphi_N \circ f)(m) = [(f \otimes id_C) \circ \varphi_M](m)$$

$$f(m)_0 \otimes f(m)_1 = f(m_0) \otimes m_1.$$

The category which objects are right C-comodules and which morphisms are right C-colinear maps is denoted $\mathscr{M}^C$. Let M and N be two objects of $\mathscr{M}^C$. Let $Hom^C(M,N)$ be the R-module of C-colinear maps from M to N.

Remark 1.5.5. *We can also define a left C-comodule M with the left coaction defined by:*

$$\varphi_M(m) = m_{-1} \otimes m_0 \in C \otimes M.$$

The category which objects are left C-comodules and which morphisms are left C-colinear maps is denoted ${}^C\mathscr{M}$.

Definition 1.5.6. *Let C be a coalgebra containing a grouplike element x. Let M be a right C-comodule. An element $m \in M$ is said to be (C,x)-coinvariant if*

$$\varphi_M(m) = m \otimes x.$$

The set of (C,x)-coinvariant elements of M is denoted $M^{coC,x}$; that is,

$$M^{coC,x} = \{m \in M; \varphi_M(m) = m \otimes x\}.$$

$M^{coC,x}$ is a sub-R-module of M called a submodule of the coinvariants of M.

Definition 1.5.7. *Let H be a Hopf algebra and let M be a right H-comodule. An element m of M is said to be H-coinvariant if*

$$\varphi_M(m) = m \otimes 1_H.$$

We set

$$M^{coH} = \{m \in M; \varphi_M(m) = m \otimes 1_H\}$$

the set of H-coinvariant elements of M. It is also a sub-R-module of M.

An R-module M is a trivial H-comodule if M is an H-comodule and all its elements are H-coinvariant.

Theorem 1.5.8. *Let C be a coalgebra and M be a right C-comodule. Then M is a left C^*-module with the action defined by:*

$$f.m = f(m_1)m_0, \quad \forall f \in C^*, m \in M.$$

Definition 1.5.9. ***(Hopf H-modules)***
Let H be a Hopf algebra. An R-module M is a right Hopf H-module if it is simultaneously:

- *a right H-module,*
- *a right H-comodule,*

such that:

$$(mh)_0 \otimes (mh)_1 = m_0h_1 \otimes m_1h_2, \quad \forall m \in M, h \in H.$$

We denote by $\mathscr{M}_H^H$ the category of right Hopf H-modules. The objects are the right Hopf H-modules and the morphisms are the morphisms of Hopf H-modules; that is, morphisms which are simultaneously morphisms of right H-modules and morphisms of right H-comodules. Analogously, we define:

- *${}_H\mathscr{M}^H$ the category of left right Hopf H-modules,*
- *${}^H\mathscr{M}^H$ the category of right Hopf H-modules,*
- *${}_H^H\mathscr{M}$ the category of left Hopf H-modules.*

Lemma 1.5.10. *Let H be a Hopf algebra.*

1) *H is a Hopf H-module with the action defined by the multiplication and the coaction by the coproduct.*

2) *Let M be a Hopf H-module. Then $M^{coH} \otimes H$ is a right Hopf H-module with the action:*

$$(m \otimes h)h' = m \otimes (hh'), \quad m \in M, \quad h, h' \in H$$

and the coaction:

$$\varphi_{M^{coH} \otimes H}(m \otimes h) = m \otimes h_1 \otimes h_2.$$

Theorem 1.5.11. ***(Fundamental theorem of Hopf H-modules)***
Let R be a field and H a Hopf algebra with antipode S_H. Let M be a right Hopf H-module; the map

$$\chi : M^{coH} \otimes H \longrightarrow M, \quad m \otimes h \longmapsto mh$$

is an isomorphism of right Hopf H-modules.

Definition 1.5.12. ***(H-comodule algebras)***
Let H be a Hopf algebra and A an R-algebra. We say that A is a right $H-$comodule algebra if A is a right H-comodule such that:

$$(ab)_0 \otimes (ab)_1 = a_0b_0 \otimes a_1b_1 \quad and \quad \varphi_A(1_A) = 1_A \otimes 1_H.$$

Definition 1.5.13. ***(relative Hopf modules)***
Let H be a Hopf algebra and A be a right H-comodule algebra. An R-module M is a relative right Hopf (A,H)-module, if M is:

- *a right A-module,*
- *a right H-comodule, such that:*

$$(ma)_0 \otimes (ma)_1 = m_0 a_0 \otimes m_1 a_1.$$

We denote by $\mathscr{M}_A^H$ the category of relative right Hopf (A,H)-modules. The objects are the relative right Hopf (A,H)-modules and the morphisms are the morphisms between relative right Hopf (A,H)-modules, that is, the maps which are simultaneously right A-linear and right H-colinear.
Analogously, we define:

- ${}_A\mathscr{M}^H$ the category relative left right Hopf (A,H)-modules
- ${}^H\mathscr{M}_A$ the category relative right left Hopf (A,H)-modules
- ${}_A^H\mathscr{M}$ the category relative left Hopf (A,H)-modules

Definition 1.5.14. *Let M and N be two relative right Hopf (A,H)-modules. We denote $Hom_A^H(M,N)$, the R-module of R-homomorphisms $f : M \longrightarrow N$ which are simultaneously A-linear and H-colinear maps; in other words,*

$$Hom_A^H(M,N) = \{f \in Hom_A(M,N) \cap Hom^H(M,N) / M,N \in \mathscr{M}_A^H\}.$$

Remark 1.5.15. *Let M be a right R-module. The R-module $End_R(M)$ equipped with the usual addition, the multiplication by a scalar and composition of maps is an algebra.*

1.6 Monoidal, braided monoidal, and symmetric monoidal categories

In this part, our goal is not to develop the theory of categories but just to state some notions and useful results for the rest of this document. So for all generalities on category theory, see [54], [22, p. 361], [9], [52], [2], [7], $\cdots$

Definition 1.6.1. ***(cf.[52, Sect. 1.2])***
A category $\mathscr{C}$ consists of three ingredients: a class $Obj(\mathscr{C})$ of ***objects****, a set of* ***morphisms*** *$Hom(M,N)$ for every ordered pair (M,N) of objects, and* ***composition***

$$Hom(M,N) \times Hom(N,P) \to Hom(M,P),$$

denoted by

$$(f,g) \mapsto g \circ f,$$

for every ordered triple M,N,P of objects. These ingredients are subject to the following axioms:

1. *the Hom sets are pairwise disjoint; that is, each $f \in Hom(M,N)$ has a unique* ***domain*** *M and a unique* ***target*** *N;*

2. *for each object M, there is an* ***identity morphism*** *$1_M \in Hom(M,M)$ such that $f \circ 1_M = f$ and $1_N \circ f = f$ for all $f : M \to N$,*

3. *composition is associative: given morphisms* $M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} Q,$ *then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Definition 1.6.2. ***(subcategory, cf.[52, Sect. 1.2])***
A category $\mathscr{D}$ is a subcategory of a category $\mathscr{C}$ if:

1. $Obj(\mathscr{D}) \subseteq Obj(\mathscr{C})$,

2. $Hom_{\mathscr{D}}(M,N) \subseteq Hom_{\mathscr{C}}(M,N)$ *for all $M,N \in Obj(\mathscr{D})$, where we denote Hom sets in $\mathscr{D}$ by $Hom_{\mathscr{D}}(-,-)$,*

3. *if $f \in Hom_{\mathscr{D}}(M,N)$ and $g \in Hom_{\mathscr{D}}(N,P)$, then the composite $g \circ f \in Hom_{\mathscr{D}}(M,P)$ is equal to the composite $g \circ f \in Hom_{\mathscr{C}}(M,P)$,*

4. *if $M \in Obj(\mathscr{D})$, then the identity $1_M \in Hom_{\mathscr{D}}(M,M)$ is equal to the identity $1_M \in Hom_{\mathscr{C}}(M,M)$.*

A subcategory $\mathscr{D}$ of $\mathscr{C}$ is ***a full subcategory*** *if, for all $M,N \in Obj(\mathscr{D})$, we have $Hom_{\mathscr{D}}(M,N) = Hom_{\mathscr{C}}(M,N)$.*

Definition 1.6.3. ***(Dual category)*** *A category $\mathscr{C}'$ is said to be a* ***dual category*** *of the category $\mathscr{C}$ if:*

1. $Obj(\mathscr{C}) = Obj(\mathscr{C}')$,

2. $Hom_{\mathscr{C}}(M,N) = Hom_{\mathscr{C}'}(N,M)$.

Definition 1.6.4. ***(Covariant functor)***
Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. The functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is a function such that:

- *if $M \in Obj(\mathscr{C})$, then $\mathscr{F}(M) \in Obj(\mathscr{D})$,*

- *if $f : M \longrightarrow M'$ in $\mathscr{C}$, then* $\mathscr{F}(f) : \mathscr{F}(M') \longrightarrow \mathscr{F}(M)$ *in $\mathscr{D}$ (note the reversal of arrows),*

- *if* $M \xrightarrow{f} M' \xrightarrow{g} M''$ *in $\mathscr{C}$, then* $\mathscr{F}(M'') \xrightarrow{\mathscr{F}(g)} \mathscr{F}(M') \xrightarrow{\mathscr{F}(f)} \mathscr{F}(M)$ *and*

$$\mathscr{F}(g \circ f) = \mathscr{F}(f) \circ \mathscr{F}(g),$$

- $\mathscr{F}(1_M) = 1_{\mathscr{F}(A)}$ *for every $M \in Obj(\mathscr{C})$.*

The functor $\mathscr{F}$ defined above is called ***covariant functor***.

Definition 1.6.5. ***(contravariant functor)***

Let $\mathscr{C}'$ (respectively $\mathscr{D}'$) be the dual category to the category $\mathscr{C}$ (respectively $\mathscr{D}$). A covariant functor from the dual category $\mathscr{C}'$ to $\mathscr{D}$ (or $\mathscr{D}'$ to $\mathscr{C}$) called a ***contravariant functor*** *from $\mathscr{C}$ to $\mathscr{D}$.*

Remark 1.6.6.

- *If $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is a covariant functor, then for any object $M,N\in\mathscr{C}$, we have a map*

$$Hom_{\mathscr{C}}(M,N)\longrightarrow Hom_{\mathscr{D}}(\mathscr{F}(M),\mathscr{F}(N)) \quad \text{given by} \quad f\longmapsto \mathscr{F}(f).$$

- *If this map is* ***injective (surjective, bijective)****, then the functor $\mathscr{F}$ is called* ***faithful (full, full and faithful)****.*

Definition 1.6.7. ***(Functorial morphisms or natural transformations)***
Let $\mathscr{F},\mathscr{G}:\mathscr{C}\longrightarrow\mathscr{D}$ be two functors. A ***functorial morphism*** *$\phi:\mathscr{F}\longrightarrow\mathscr{G}$ is a family $\{\phi(M)\ /\ M\in\mathscr{C}\}$ of morphisms, such that*

$$\phi(M):\mathscr{F}(M)\longrightarrow\mathscr{G}(M)\ \ \forall M\in\mathscr{C}$$

and for any morphism $f:M\longrightarrow N$ in $\mathscr{C}$, we have that

$$\phi(N)\circ\mathscr{F}(f)=\mathscr{G}(f)\circ\phi(M).$$

If moreover $\phi(M)$ is an isomorphism for any $M\in\mathscr{C}$, then ϕ is called a ***functorial isomorphism.*** *If there exists such a functorial isomorphism we write $\mathscr{F}\cong\mathscr{G}$.*

Definition 1.6.8. ***(Adjoint pair)***
Let $\mathscr{F},\mathscr{G}:\mathscr{C}\longrightarrow\mathscr{D}$ and $\mathscr{G}:\mathscr{D}\longrightarrow\mathscr{C}$ be two covariant functors. The ordered pair $(\mathscr{F},\mathscr{G})$ is an ***adjoint pair*** *if, for each $M\in Obj(\mathscr{C})$ and $N\in Obj(\mathscr{D})$, there are bijections*

$$\phi:Hom_{\mathscr{D}}(\mathscr{F}(M),N)\longrightarrow Hom_{\mathscr{C}}(M,\mathscr{G}(N)),$$

that are natural transformations in $\mathscr{C}$ and in $\mathscr{D}$.

Definition 1.6.9. ***(Equivalence of categories)***
Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A covariant functor $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is called an equivalence of categories if there exists a covariant functor $\mathscr{G}:\mathscr{D}\to\mathscr{C}$ such that $\mathscr{F}\circ\mathscr{G}\cong 1_{\mathscr{D}}$ and $\mathscr{G}\circ\mathscr{F}\cong 1_{\mathscr{C}}$, where $1_{\mathscr{C}}:\mathscr{C}\to\mathscr{C}$ is the identity functor and it is defined by $1_{\mathscr{C}}(M)=M$ for any object M and $1_{\mathscr{C}}(f)=f$ for any morphism f. If moreover $\mathscr{F}\circ\mathscr{G}=1_{\mathscr{D}}$ and $\mathscr{G}\circ\mathscr{F}=1_{\mathscr{C}}$, then $\mathscr{F}$ is called an isomorphism of categories, and $\mathscr{C}$ and $\mathscr{D}$ are called isomorphic categories.

Theorem 1.6.10. ***([22, Theorem A.2.1])***
If $\mathscr{F}:\mathscr{C}\longrightarrow\mathscr{D}$ is a covariant functor, then $\mathscr{F}$ is an equivalence of categories if and only if the following conditions are satisfied.

1. *$\mathscr{F}$ is a full and faithful functor.*

2. *For any object $N \in \mathscr{D}$ there exists an object $M \in \mathscr{C}$ such that $N \cong \mathscr{F}(M)$.*

A contravariant functor $\mathscr{F} : \mathscr{C} \longrightarrow \mathscr{D}$ such that $\mathscr{F}$ is an equivalence between $\mathscr{C}'$ (the dual of $\mathscr{C}$) and $\mathscr{D}$ (or $\mathscr{C}$ and $\mathscr{D}'$ (the dual of $\mathscr{D}$)) is called **duality**.

Definition 1.6.11. ***(Monoidal category)***
A monoidal category (or tensor category) $\mathscr{C} = (\mathscr{C}, \otimes, I, \alpha, l, r)$ consists of:

- *a category $\mathscr{C}$,*
- *a functor called the tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ such that $(M,N) \mapsto M \otimes N$ and $(f,g) \mapsto f \otimes g$ for all objects $M,N \in \mathscr{C}$ and morphisms $f,g \in \mathscr{C}$,*
- *an object called the identity object $I \in \mathscr{C}$,*
- *natural isomorphisms called,*
  the associator:
  $$\alpha_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$$
  the left unit law:
  $$l_M : I \otimes M \to M$$
  and the right unit law:
  $$r_M : M \otimes I \to M$$
  such that the following diagrams commute for all objects $M,N,P,Q \in \mathscr{C}$, we have:

  - *the pentagon diagram:*

$$\begin{tikzcd}
((M \otimes N) \otimes P) \otimes Q \arrow[r, "\alpha_{M,N,P} \otimes id_Q"] \arrow[d, "\alpha_{M\otimes N,P,Q}"'] & (M \otimes (N \otimes P)) \otimes Q \arrow[r, "\alpha_{M,N\otimes P,Q}"] & M \otimes ((N \otimes P) \otimes Q) \arrow[d, "id_M \otimes \alpha_{N,P,Q}"] \\
(M \otimes N) \otimes (P \otimes Q) \arrow[rr, "\alpha_{M,N,P\otimes Q}"'] & & M \otimes (N \otimes (P \otimes Q))
\end{tikzcd}$$

  - *the triangle diagram:*

$$\begin{tikzcd}
(M \otimes I) \otimes N \arrow[rr, "\alpha_{M,I,N}"] \arrow[dr, "r_M \otimes id_N"'] & & M \otimes (I \otimes N) \arrow[dl, "id_M \otimes l_N"] \\
& M \otimes N &
\end{tikzcd}$$

Definition 1.6.12. ***(Monoidal functor)***
Let $\mathscr{C} = (\mathscr{C}, \otimes, I, \alpha_{\mathscr{C}}, l_{\mathscr{C}}, r_{\mathscr{C}})$ and $\mathscr{D} = (\mathscr{D}, \otimes, J, \alpha_{\mathscr{D}}, l_{\mathscr{D}}, r_{\mathscr{D}})$ be two monoidal categories. A monoidal functor between $\mathscr{C}$ and $\mathscr{D}$ is the triplet $(\mathscr{F}, \varphi_0, \varphi_2)$ such that $\mathscr{F} : \mathscr{C} \longrightarrow \mathscr{D}$ is a functor and

$$\varphi_0 : J \longrightarrow \mathscr{F}(I), \qquad \varphi_2 : \mathscr{F}(-) \otimes \mathscr{F}(-) \longrightarrow \mathscr{F}(- \otimes -)$$

are functorial isomorphisms such that, for all objects $M,N,P \in \mathscr{C}$, the following diagrams are commutative:

$$\begin{tikzcd}
(\mathscr{F}(M)\otimes\mathscr{F}(N))\otimes\mathscr{F}(P) \arrow[r, "\alpha_{\mathscr{D}}"] \arrow[d, "\varphi_2(M,N)\otimes id_{\mathscr{F}(P)}"'] & \mathscr{F}(M)\otimes(\mathscr{F}(N)\otimes\mathscr{F}(P)) \arrow[d, "id_{\mathscr{F}(M)}\otimes\varphi_2(N,P)"] \\
\mathscr{F}(M\otimes N)\otimes\mathscr{F}(P) \arrow[d, "\varphi_2(M\otimes N,P)"'] & \mathscr{F}(M)\otimes\mathscr{F}(N\otimes P) \arrow[d, "\varphi_2(M,N\otimes P)"] \\
\mathscr{F}((M\otimes N)\otimes P) \arrow[r, "\mathscr{F}(\alpha_{\mathscr{C}})"'] & \mathscr{F}(M\otimes(N\otimes P))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{F}(M)\otimes J \arrow[r, "id_{\mathscr{F}(M)}\otimes\varphi_0"] \arrow[d, "r_{\mathscr{F}(M)}"'] & \mathscr{F}(M)\otimes\mathscr{F}(I) \arrow[d, "\varphi_2(M,I)"] \\
\mathscr{F}(M) & \mathscr{F}(M\otimes I) \arrow[l, "\mathscr{F}(r_M)"]
\end{tikzcd}$$

$$\begin{tikzcd}
J\otimes\mathscr{F}(M) \arrow[r, "\varphi_0\otimes id_{\mathscr{F}(M)}"] \arrow[d, "l_{\mathscr{F}(M)}"'] & \mathscr{F}(I)\otimes\mathscr{F}(M) \arrow[d, "\phi_2(I,M)"] \\
\mathscr{F}(M) & \mathscr{F}(I\otimes M) \arrow[l, "\mathscr{F}(l_M)"]
\end{tikzcd}$$

Definition 1.6.13. ***(braided monoidal category)***
A braided monoidal category consists of:

- *a monoidal category $\mathscr{C}$,*
- *a natural isomorphism called the braiding:*

$$\gamma_{M,N} : M\otimes N \longrightarrow N\otimes M$$

such that these two diagrams commute, called the hexagon diagrams:

$$\begin{tikzcd}
 & M\otimes(N\otimes P) \arrow[r, "\gamma"] & (N\otimes P)\otimes M \arrow[dr, "\alpha"] & \\
(M\otimes N)\otimes P \arrow[ur, "\alpha"] \arrow[dr, "\gamma\otimes id_P"'] & & & N\otimes(P\otimes M) \\
 & (N\otimes M)\otimes P \arrow[r, "\alpha"'] & N\otimes(M\otimes P) \arrow[ur, "id_N\otimes\gamma"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (M\otimes N)\otimes P \arrow[r, "\gamma"] & P\otimes(M\otimes N) \arrow[dr, "\alpha^{-1}"] & \\
M\otimes(N\otimes P) \arrow[ur, "\alpha^{-1}"] \arrow[dr, "id_M\otimes\gamma"'] & & & (P\otimes M)\otimes N \\
& M\otimes(P\otimes N) \arrow[r, "\alpha^{-1}"'] & (M\otimes P)\otimes N \arrow[ur, "\gamma\otimes id_N"'] &
\end{tikzcd}$$

Definition 1.6.14. ***(braided monoidal functor)***
Let $\mathscr{C}$ and $\mathscr{D}$ be two braided monoidal categories. A functor $\mathscr{F}:\mathscr{C}\longrightarrow\mathscr{D}$ is braided monoidal if it is monoidal and it makes the following diagram commutative for all objects N and M

$$\begin{tikzcd}
\mathscr{F}(M)\otimes\mathscr{F}(N) \arrow[r, "\gamma_{\mathscr{F}(M),\mathscr{F}(N)}"] \arrow[d, "\varphi_2(M,N)"'] & \mathscr{F}(N)\otimes\mathscr{F}(M) \arrow[d, "\varphi_2(N,M)"] \\
\mathscr{F}(M\otimes N) \arrow[r, "\mathscr{F}(\gamma_{M,N})"'] & \mathscr{F}(N\otimes M)
\end{tikzcd}$$

Definition 1.6.15. ***(symmetric monoidal category)***
A symmetric monoidal category is a braided monoidal category $\mathscr{C}$ for which the braiding satisfies

$$\gamma_{M,N}=\gamma_{N,M}^{-1};$$

for all objects M and N of $\mathscr{C}$.

A monoidal, braided monoidal, or symmetric monoidal category is called strict if α, l, r are all identity morphisms.

Chapter 2

A BRAUER-CLIFFORD-LONG GROUP FOR THE CATEGORY OF DYSLECTIC (S,H)-DIMODULE ALGEBRAS

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Abstract

In this article, we define the notion of Brauer-Clifford group for the category of dyslectic (S,H)-dimodules, where H is a commutative and cocommutative Hopf algebra and S is an H-commutative dimodule algebra. This Brauer group turns out to be an example of the Brauer group of a braided monoidal category. We also show that this Brauer group is anti-isomorphic to the Brauer group of the category of dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-modules.

Introduction

The classical notion of Brauer group was introduced by Richard Brauer in his study of arithmetic field theory before being generalized by Azumaya in [8] and Auslander-Goldman in [6] respectively in the case of the Brauer group of a local base ring and the Brauer group of a general commutative base ring. This notion is later extended by Auslander in [5] to the case of a topological space endowed with its structural sheaf of rings of continuous complex valued functions. Grothendieck showed in [26] that the set of isomorphic classes of Azumaya algebras with constant rank n^2 over a topological space, is identified with the set of isomorphic classes of $GP(n)$-principal bundles with base this space.
The isomorphic classes of Azumaya algebras over a commutative ring R form a set which in turn is a group $B(R)$ (with multiplication given by the tensor product over R) known in the literature as the Brauer group of R. For symmetric monoidal categories, Brauer groups were defined by Pareigis [47] and more generally for the braided monoidal categories by Van Oys-

taeyen and Zhang [59]. Studies have also been developed in this same context by [24], [60], [23]. Recently, in [4], Ardizzoni and El Kaoutit give a characterization of the automorphism group of an Azumaya comodule algebra over a commutative flat Hopf algebroid.

A braided monoidal category is a category $\mathscr{C}$ with a tensor product $\otimes$ that has a unit object R and has a family of isomorphisms $\gamma_{M,N} : M \otimes N \longrightarrow N \otimes M$, one for each pair of objects in $\mathscr{C}$, satisfying natural coherence conditions. If $\gamma_{N,M} \circ \gamma_{M,N} = id_{M\otimes N}$ always holds, then the braided monoidal category is said to be symmetric. If $\mathscr{C}$ is a braided monoidal category, we know from [59] that we can define a Brauer group in $\mathscr{C}$.

Brauer-Clifford groups are equivariant Brauer groups for which a Hopf algebra acts or coacts non-trivially on a certain commutative base ring.

In [40], Long introduced a Brauer group $BD(R,H)$ of H-dimodule algebras (that is, algebras with an H-action and an H-coaction satisfying some compatibility conditions), where R is a commutative ring and H is a commutative and cocommutative Hopf algebra over R. This Brauer group is called Long's Brauer group. The category $\mathscr{D}^H$ of H-dimodules is a braided monoidal category: the braiding γ is defined by

$$\gamma_{M,N} : M \otimes_R N \longrightarrow N \otimes_R M; \quad m \otimes n \mapsto m_1 n \otimes m_0.$$

In [17], Caenepeel, Van Oystaeyen, and Zhang generalized Long's construction to Yetter-Drinfel'd's modules also called quantum Yang-Baxter modules and to Yetter-Drinfel'd module algebras, where H is a Hopf algebra with a bijective antipode. The category $\mathscr{Q}^H$ of Yetter-Drinfel'd modules is a braided monoidal category: the braiding γ is defined by

$$\gamma_{M,N} : M \otimes_R N \longrightarrow N \otimes_R M; m \otimes n \mapsto n_0 \otimes n_1 m.$$

They defined the Brauer group $BQ(R,H)$ of Hopf Yetter-Drinfel'd H-modules Azumaya algebras. In the same paper, the authors introduced (Proposition: 2.1) the category ${}_S\mathscr{Q}^H$ of Hopf Yetter-Drinfel'd (S,H)-modules, where S is H-commutative with respect to the braiding of $\mathscr{Q}^H$. They showed that ${}_S\mathscr{Q}^H$ is a monoidal category.
In [32], Guédénon and Herman considered the abelian full subcategory $Dys\text{-}{}_S\mathscr{Q}^H$ of dyslectic Hopf Yetter-Drinfel'd (S,H)-modules adapting a conditon of [50]. The authors showed that $Dys\text{-}{}_S\mathscr{Q}^H$ is a braided monoidal category: the braiding is the extension of the braiding of $\mathscr{Q}^H$ to $Dys\text{-}{}_S\mathscr{Q}^H$ and they defined a Brauer group $BQ(S,H)$ of Azumaya algebras in $Dys\text{-}{}_S\mathscr{Q}^H$ called the Brauer-Clifford-Long group of dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebras. When H is commutative and cocommutative, we get from [32] the Brauer-Clifford-Long group of dyslectic (S,H)-dimodule Azumaya algebras.

In the present paper, we introduce the notion of (dyslectic) (S,H)-dimodules, where H is commutative and cocommutative and S is H-commutative with respect to the braiding of $\mathscr{D}^H$. We denote by ${}_S\mathscr{D}^H$ the category of (S,H)-dimodules and by $Dys\text{-}{}_S\mathscr{D}^H$ its subcategory of dyslectic (S,H)-dimodules. We show that $Dys\text{-}{}_S\mathscr{D}^H$ is a braided monoidal category: the braiding is induced by the braiding of $\mathscr{D}^H$. We define algebras and Azumaya algebras in $Dys\text{-}{}_S\mathscr{D}^H$. From that, we define the Brauer group $BD(S,H)$ of Azumaya algebras in $Dys\text{-}{}_S\mathscr{D}^H$ which we call the Brauer-Clifford-Long group of dyslectic (S,H)-dimodules Azumaya algebras. Our result is

a direct generalization of Long's Brauer group. In particular, we get a generalization of the Brauer-Clifford group $B(S,H)$ of (S,H)-Azuamaya algebras, when H is cocommutative and S is commutative, and a generalization of the Brauer-Clifford group $B^{co}(S,H)$ of Hopf (S,H)-Azumaya algebras studied in [31], when H is commutative and S commutative.
We also establish an anti-isomorphism of groups between our Brauer-Clifford-Long group $BD(S,H)$ and the Brauer-Clifford-Long group $BQ(S^{op},H)$ of algebras in the category of Hopf Yetter-Drinfeld (S^{op},H)-modules studied in [32], where S^{op} is the opposite algebra of S. This last result is a generalization of the well-known result which asserts that the Brauer group of Long dimodules and the Brauer group of Yetter-Drinfeld modules are anti-isomorphic if H is commutative and cocommutative.

The paper is organized as follows. In Section 1, we review some results of Hopf algebras and some preliminary results concerning dimodules and braided monoidal categories.

In Section 2, we introduce the notion of (S,H)-dimodules, where S is an H-dimodule algebra and we define the category ${}_S\mathscr{D}^H$ of (S,H)-dimodules with (S,H)-dimodule homomorphisms.

In Section 3, we introduce the concept of dyslectic (S,H)-dimodules and we define the category $Dys\text{-}{}_S\mathscr{D}^H$ of dyslectic (S,H)-dimodules in order to obtain a braided monoidal category. The monoidal structure is given by relative tensor products over S. Note that the category $Dys\text{-}{}_S\mathscr{D}^H$ is an abelian full subcategory of ${}_S\mathscr{D}^H$.

In Sections 4 and 5, we introduce the concepts of dyslectic (S,H)-dimodule algebras and dyslectic (S,H)-dimodule Azumaya algebras in the category $Dys\text{-}{}_S\mathscr{D}^H$, and we give all the required ingredients to define the Brauer-Clifford-Long group $BD(S,H)$.

In Section 6, we give some elementary homomorphisms between Brauer- Clifford-Long groups that are induced by scalar extensions and central twists.

Finally, in Section 7, we establish an anti-isomorphism of groups between the Brauer-Clifford-Long group $BD(S,H)$ of algebras in the category of dyslectic (S,H)-dimodules and the Brauer-Clifford-Long group $BQ(S^{op},H)$ of algebras in the category of dyslectic Hopf-Yetter-Drinfeld (S^{op},H)-modules, where S^{op} is the opposite algebra of S.

For more details on Hopf algebras and Brauer groups, we refer to the literature, see for example [1], [14], [44], [56]. Throughout the paper, R stands for a commutative ring with unit, and H is a Hopf algebra over R. Any algebras, modules and unadorned tensor products are always over R

2.1 Preliminaries and Notations

Let H be a Hopf R-algebra. We denote its comultiplication by $\Delta : H \to H \otimes H$, its antipode by $S_H : H \to H$ and its counit by $\varepsilon : H \to R$. We will use Sweedler-Heyneman notation, omitting sums, so we write $\Delta(h) = h_1 \otimes h_2$.

A Hopf algebra H is said to be *cocommutative* if $h_1 \otimes h_2 = h_2 \otimes h_1$, for all $h \in H$. We will require a sequence of definitions, all of which are standard. An R-algebra A is an *H-module* *algebra* if A is a left H-module such that

$$h.(ab) = (h_1.a)(h_2.b) \text{ and } h.1_A = \varepsilon(h)1_A, \text{ for all } a,b \in A, h \in H. \tag{2.1.1}$$

H acts trivially on A when $h.a = \varepsilon(h)a$ for all $h \in H$ and $a \in A$. *A homomorphism of H-module algebras* is a homomorphism of H-modules which is also a homomorphism of R-algebras. If A is an H-module algebra, then the smash product algebra $A\#H$ is the R-module $A \otimes H$ with multiplication

$$(a \otimes h)(a' \otimes h') = a(h_1.a') \otimes h_2h', \text{ for all } a,a' \in A \text{ and } h,h' \in H. \tag{2.1.2}$$

An R-module M is a left $A\#H$-module if it is a left A-module ($a \otimes m \mapsto a \rightharpoonup m$, where $\rightharpoonup$ denotes the left A-action on M) and a left H-module for which

$$h(a \rightharpoonup m) = (h_1.a) \rightharpoonup (h_2m), \text{ for all } h \in H, a \in A \text{ and } m \in M. \tag{2.1.3}$$

If A is an H-module algebra and S is a sub-H-module algebra of A, then the algebras A and S are left $S\#H$-modules. We will write ${}_{A\#H}\mathcal{M}$ for the category of left $A\#H$-modules. It was observed in [31, Theorem 2.2] that if H is cocommutative and A is a commutative H-module algebra, then $({}_{A\#H}\mathcal{M}, \otimes_A, A)$ is a symmetric monoidal category.
If H is a Hopf algebra over R, an R-module M is a *right H-comodule* if there exists an R-linear map $\rho_M : M \to M \otimes H$ satisfying $(\rho_M \otimes id_H) \circ \rho_M = (id_M \otimes \Delta) \circ \rho_M$ and $(id_M \otimes \varepsilon) \circ \rho_M = id_M$. In Sweedler notation, we write $\rho_M(m) = m_0 \otimes m_1$ for all $m \in M$, and the right H-comodule conditions on M are

$$m_{00} \otimes m_{01} \otimes m_1 = m_0 \otimes m_{11} \otimes m_{12} \text{ and } m_0\varepsilon(m_1) = m, \text{ for all } m \in M. \tag{2.1.4}$$

H coacts trivially on M when $m_0 \otimes m_1 = m_0 \otimes 1_H$, for all $m \in M$. Let M and N be right H-comodules. A *homomorphism of right H-comodules* (aka. a right H-colinear map) is an R-linear map $f : M \to N$ such that $\rho_N \circ f = (f \otimes id_H) \circ \rho_M$. In Sweedler notation, this is equivalent to

$$f(m)_0 \otimes f(m)_1 = f(m_0) \otimes m_1, \text{ for all } m \in M. \tag{2.1.5}$$

If M and N are right H-comodules, then $M \otimes N$ is a right H-comodule under the codiagonal coaction:

$$(m \otimes n)_0 \otimes (m \otimes n)_1 = m_0 \otimes n_0 \otimes m_1n_1, \quad m \in M, n \in N. \tag{2.1.6}$$

An R-algebra A is an *H-comodule algebra* if A is a right H-comodule and the multiplication in A satisfies

$$(ab)_0 \otimes (ab)_1 = a_0b_0 \otimes a_1b_1 \text{ and } \rho_A(1_A) = 1_A \otimes 1_H, \text{ for all } a,b \in A. \tag{2.1.7}$$

A *homomorphism of H-comodule algebras* is a homomorphism of H-comodules which is also a homomorphism of R-algebras.
Let A be a right H-comodule algebra. An R-module M is an *(A,H)-Hopf module* if M is both a left A-module and a right H-comodule, with the property

$$(a \rightharpoonup m)_0 \otimes (a \rightharpoonup m)_1 = (a_0 \rightharpoonup m_0) \otimes a_1m_1, \text{ for all } a \in A, m \in M. \tag{2.1.8}$$

A *homomorphism of* (A,H)*-Hopf modules* is a left A-linear map which is also a right H-colinear map. We will write ${}_A\mathscr{M}^H$ for the category of (A,H)-Hopf modules. This category is dual to ${}_{A\#H}\mathscr{M}$, and when H is commutative and A is a commutative H-comodule algebra, $({}_A\mathscr{M}^H, \otimes_A, A)$ is a symmetric monoidal category.

Let H be commutative and cocommutative. An *H-dimodule* is an R-module M which is a left H-module and a right H-comodule with H-structures maps $\lambda_M : H\otimes M \to M$ and $\rho_M : M \to M\otimes H$ such that $\rho_M \circ \lambda_M = (\lambda_M \otimes id_M)\circ(id_H \otimes \rho_M)$, that is:

$$(hm)_0 \otimes (hm)_1 = hm_0 \otimes m_1, \quad \forall h \in H, m \in M. \tag{2.1.9}$$

If M and N are H-dimodules, an R-linear map $f : M \to N$ is said to be an *H-dimodule homomorphism* if it is simultaneously an H-module homomorphism and an H-comodule homomorphism.
An *H-dimodule algebra* is an R-algebra which is an H-dimodule so that it is an H-module algebra and an H-comodule algebra satisfying the relation (2.1.9).

An *H-dimodule algebra homomorphism* between two H-dimodule algebras A and B is an R-linear map $A \to B$ which is simultaneously an H-dimodule homomorphism and an R-algebra homomorphism.
We denote the category of H-dimodules by $\mathscr{D}^H$. For H-dimodules M and N, the tensor product $M\otimes N$ has an H-module structure given by

$$h(m\otimes n) = (h_1 m)\otimes(h_2 n), \quad \forall m \in M, n \in N, \tag{2.1.10}$$

and an H-comodule structure given by

$$(m\otimes n)_0 \otimes (m\otimes n)_1 = m_0 \otimes n_0 \otimes m_1 n_1, \quad \forall m \in M, n \in N. \tag{2.1.11}$$

These H-structures satisfy the compatibility condition (2.1.9) and make of $M\otimes N$ an H-dimodule, denoted by $M\tilde{\otimes}N$.
Since H is commutative and cocommutative, for H-dimodules M and N, there exists an H-dimodule isomorphism $\gamma_{M,N}$ from $M\tilde{\otimes}N$ to $N\tilde{\otimes}M$ defined by (see [59])

$$\gamma_{M,N}(m\tilde{\otimes}n) = m_1.n\tilde{\otimes}m_0, \quad \forall m \in M, n \in N, \tag{2.1.12}$$

with inverse

$$\gamma^{-1}_{M,N}(n\tilde{\otimes}m) = m_0\tilde{\otimes}S_H(m_1)n, \quad \forall m \in M, n \in N. \tag{2.1.13}$$

According to [59, Example 3.11], $(\mathscr{D}^H, \tilde{\otimes}_R, \gamma_{M,N}, R)$ is a braided monoidal category. Note that an H-dimodule algebra is just an algebra in the braided monoidal category $\mathscr{D}^H$. Recall that a monoidal category $(\mathscr{C}, \otimes, R)$ is braided if there are natural isomorphisms $\gamma_{M,N} : M\otimes N \cong N\otimes M$ in $\mathscr{C}$, for all $M,N \in \mathscr{C}$, such that the following hexagonal coherence conditions are satisfied (see [42, p. 180])

$$\gamma_{M\otimes N,P} = (\gamma_{M,P}\otimes 1)\circ(1\otimes\gamma_{N,P}) \text{ and } \gamma_{M,N\otimes P} = (1\otimes\gamma_{M,P})\circ(\gamma_{M,N}\otimes 1), \text{ for all } M,N,P \in \mathscr{C}.$$

Let's give two examples of H-dimodules.

Example 2.1.1. *H is a group algebra RG.*

Let G be a multiplicative finite abelian group with unit element e. The group algebra RG is a commutative and cocommutative Hopf algebra. A G-graded R-module M is an R-module M with a fixed decomposition $M = \bigoplus_{\sigma\in G} M_\sigma$, where each M_σ is an R-submodule of M. M_σ is the set of the homogeneous elements of degree σ. Every element of M is a finite sum of homogeneous elements. It is well known that an R-module is a G-graded R-module if and only if it is an RG-comodule. According to [15], a G-dimodule is a left G-module which is also a G-graded R-module and the G-action is compatible with the gradation, that is, $\sigma'.M_\sigma \subseteq M_\sigma$, for all $\sigma,\sigma' \in G$.

We denote by $\mathscr{D}^G$ the category of G-dimodules: its morphisms are the graded R-linear maps of degree e which are G-linear. It is well known that the categories $\mathscr{D}^G$ and $\mathscr{D}^{RG}$ are isomorphic.

Let's consider the particular case $G = \{e,\sigma\}$, the cyclic group of order 2 (see [45]).
Let M be an RG-dimodule. The coaction of RG on M is

$$\rho_M(m) = m_{(0)} \otimes e + m_{(1)} \otimes \sigma, \text{ where } m_{(0)}, m_{(1)} \in M.$$

We have: $m = m_{(0)} + m_{(1)}, \quad \forall m \in M$. The dimodule condition is

$$\rho(\sigma m) = (\sigma m_{(0)}) \otimes e + (\sigma m_{(1)}) \otimes \sigma,$$

that is: $(\sigma m)_{(0)} = \sigma m_{(0)}$ and $(\sigma m)_{(1)} = \sigma m_{(1)}$.

If we consider M as a G-dimodule $M = M_e + M_\sigma$, every element of M is a sum of an element of degree e and an element of degree σ. It follows that $m_{(0)}$ and $m_{(1)}$ are homogeneous elements of degree e and σ, respectively.

Example 2.1.2.

Let G and G' be two abelian finite groups and $\Bbbk$ a field. Set $H = Maps(G \uplus G', \Bbbk)$ the set of all maps from the disjoint union $G \uplus G'$ of G and G' to $\Bbbk$. Then $H = K \times L$, where $K = Maps(G, \Bbbk)$ and $L = Maps(G', \Bbbk)$. We claim that K and L are commutative and cocommutative Hopf algebras over $\Bbbk$ and that $H = K \times L$ is a commutative and cocommutative Hopf algebra over $R = \Bbbk \times \Bbbk$ (with component-wise operations). Therefore we can define the categories $\mathscr{D}^K$, $\mathscr{D}^L$ and $\mathscr{D}^H$.

Let M be a vector space over $\Bbbk$. So M can be seen as a left R-module via the first projection

$$R \longrightarrow \Bbbk; (\lambda, \lambda') \mapsto \lambda.$$

Suppose that M is a left K-module via

$$(k \otimes_{\Bbbk} m) \mapsto km$$

for $k \in K$ and $m \in M$, then M is a left H-module via

$$((k,l) \otimes_R m) \mapsto km,$$

for $(k,l) \in H$, $m \in M$. In the same way, if M is a right K-comodule via

$$\rho_K(m) = m_0 \otimes_{\Bbbk} m_1,$$

then M is a right H-comodule via

$$\rho_H(m) = m_{[0]} \otimes_R m_{[1]} = m_0 \otimes_R (m_1, 0)$$

From this, every K-dimodule M can be viewed as an H-dimodule: the H-dimodule condition is

$$[(k,l)m]_{[0]} \otimes_R [(k,l)m]_{[1]} = [(k,l)m_{[0]}] \otimes_R m_{[1]}, \ \forall (k,l) \in H, m \in M.$$

2.2 The category of (S,H)-dimodules

Let S be an H-dimodule algebra. An (S,H)-dimodule M is a left S-module and an H-dimodule satisfying the compatibility conditions (2.1.3) and (2.1.8), equivalently, M is a left $S\#H$-module and an (S,H)-Hopf module for which relation (2.1.9) is satisfied. An (R,H)-dimodule is just an H-dimodule. Furthermore, note that if S is an H-dimodule algebra, then S is an (S,H)-dimodule: the left action of S is given by $s \rightharpoonup s' = ss'$, for all $s, s' \in S$.
An (S,H)-*dimodule homomorphism* is an H-dimodule map which is also left S-linear. We denote by ${}_S\mathcal{D}^H$ the category consisting of (S,H)-dimodules and (S,H)-dimodule homomorphisms, that is, the left $S\#H$-linear right H-colinear maps.
If M and N are (S,H)-dimodules, we denote by ${}_{S\#H}Hom^H(M,N)$ the R-module of left $S\#H$-linear right H-colinear maps from M to N.

Let S be an H-module algebra. We say that S is H-*commutative* if

$$ss' = (s_1.s')s_0, \text{ for all } s, s' \in S. \tag{2.2.1}$$

If S is an H-commutative H-dimodule algebra, then for every left S-action on $M \in {}_S\mathcal{D}^H$ there is a corresponding right S-action defined by

$$m \leftharpoonup s = (m_1.s) \rightharpoonup m_0, \text{ for all } s \in S, m \in M. \tag{2.2.2}$$

This allows us to view M as an S-S-bimodule. Note that the left S-action and the right S-action are also related by

$$s \rightharpoonup m = m_0 \leftharpoonup (S_H(m_1).s), \text{ for all } s \in S, m \in M. \tag{2.2.3}$$

Note also that we have

$$h(m \leftharpoonup s) = (h_1 m) \leftharpoonup (h_2.s) \tag{2.2.4}$$

and

$$(m \leftharpoonup s)_0 \otimes (m \leftharpoonup s)_1 = (m_0 \leftharpoonup s_0) \otimes m_1 s_1, \text{ for all } h \in H, m \in M, s \in S. \tag{2.2.5}$$

Let S be an H-commutative H-dimodule algebra. Then for M and N in ${}_S\mathscr{D}^H$, we can endow the tensor product $M \otimes_S N$ with the following S-action and H-structures:

$$s \rightharpoonup (m\tilde{\otimes}n) = (s \rightharpoonup m)\tilde{\otimes}n, \tag{2.2.6}$$

$$h(m\tilde{\otimes}n) = h_1 m\tilde{\otimes}h_2 n \tag{2.2.7}$$

and

$$(m\tilde{\otimes}n)_0 \otimes (m \otimes n)_1 = m_0\tilde{\otimes}n_0 \otimes m_1 n_1, \tag{2.2.8}$$

for all $h \in H, s \in S, m \in M,$ and $n \in N,$ where $m\tilde{\otimes}n = m \otimes_S n$.
Note that we have

$$(m\tilde{\otimes}n) \leftharpoonup s = m\tilde{\otimes}(n \leftharpoonup s), \text{ for all } m \in M, n \in N, s \in S. \tag{2.2.9}$$

In the remainder of this section, H is a commutative and cocommutative Hopf algebra and S is an H-commutative H-dimodule algebra.

Lemma 2.2.1. *With these structures described above, $M \otimes_S N$ is an (S,H)-dimodule which is denoted $M\tilde{\otimes}_S N$.*

Proof. For all $m, m' \in M, \quad n, n' \in N$ and $s \in S$ we have:

$$\begin{aligned}
(m \leftharpoonup s)\tilde{\otimes}n \quad &= ((m_1.s) \rightharpoonup m_0)\tilde{\otimes}n \\
&= (m_1.s) \rightharpoonup (m_0\tilde{\otimes}n) \\
&= (m_0\tilde{\otimes}n)_0 \leftharpoonup [S_H((m_0\tilde{\otimes}n)_1)(m_1.s)] \\
&= (m_{00}\tilde{\otimes}n_0) \leftharpoonup [S_H(m_{01}n_1)(m_1.s)] \\
&= (m_{00}\tilde{\otimes}n_0) \leftharpoonup [S_H(n_1)S_H(m_{01})(m_1.s)] \\
&= (m_0\tilde{\otimes}n_0) \leftharpoonup [(S_H(n_1)S_H(m_{11})m_{12}).s] \\
&= (m_0\tilde{\otimes}n_0) \leftharpoonup [S_H(n_1)\varepsilon(m_1).s] \\
&= (m_0\varepsilon(m_1)\tilde{\otimes}n_0) \leftharpoonup [S_H(n_1).s] \\
&= m\tilde{\otimes}(n_0 \leftharpoonup (S_H(n_1).s)) \\
&= m\tilde{\otimes}(s \rightharpoonup n).
\end{aligned}$$

For all $h \in H, m \in M, n \in N$ and $s \in S$, we have

$$\begin{aligned}
h((m \leftharpoonup s)\tilde{\otimes}n) \quad &= (h_1(m \leftharpoonup s))\tilde{\otimes}(h_2 n) \\
&= ((h_1 m) \leftharpoonup (h_2.s))\tilde{\otimes}(h_3 n) \\
&= (h_1 m)\tilde{\otimes}((h_2.s) \rightharpoonup (h_3 n)) \\
&= (h_1 m)\tilde{\otimes}(h_2(s \rightharpoonup n)) \\
&= h(m\tilde{\otimes}(s \rightharpoonup n))
\end{aligned}$$

and

$$\begin{aligned}
\rho((m \leftharpoonup s)\tilde{\otimes}n) &= ((m \leftharpoonup s)\tilde{\otimes}n)_0 \otimes ((m \leftharpoonup s)\tilde{\otimes}n)_1 \\
&= ((m \leftharpoonup s)_0\tilde{\otimes}n_0) \otimes ((m \leftharpoonup s)_1 n_1) \\
&= ((m_0 \leftharpoonup s_0)\tilde{\otimes}n_0) \otimes ((m_1 s_1)n_1) \\
&= (m_0\tilde{\otimes}(s_0 \rightharpoonup n_0) \otimes (m_1(s_1 n_1)) \\
&= (m_0\tilde{\otimes}(s \rightharpoonup n)_0) \otimes (m_1(s \rightharpoonup n)_1) \\
&= (m\tilde{\otimes}(s \rightharpoonup n))_0 \otimes (m\tilde{\otimes}(s \rightharpoonup n))_1 \\
&= \rho(m\tilde{\otimes}(s \rightharpoonup n)).
\end{aligned}$$

So the S-action, the left H-action and the right co-action are well defined.
$M\tilde{\otimes}_S N$ is a left S-module, for all $s, s' \in S, m \in M$ and $n \in N$, we have:

$$\begin{aligned}
(ss') \rightharpoonup (m\tilde{\otimes}n) &= ((ss') \rightharpoonup m)\tilde{\otimes}n \\
&= (s \rightharpoonup (s' \rightharpoonup m))\tilde{\otimes}n \\
&= s \rightharpoonup ((s' \rightharpoonup m)\tilde{\otimes}n) \\
&= s \rightharpoonup (s' \rightharpoonup (m\tilde{\otimes}n)).
\end{aligned}$$

It is easy to see that $M\tilde{\otimes}_S N$ is a left H-module and a right H-comodule. We will prove that the equation (2.1.9) is satisfied. We have for all $h \in H, m \in M$, and $n \in N$:

$$\begin{aligned}
[h(m\tilde{\otimes}n)]_0 \otimes [h(m\tilde{\otimes}n)]_1 &= (h_1 m\tilde{\otimes}h_2 n)_0 \otimes (h_1 m\tilde{\otimes}h_2 n)_1 \\
&= ((h_1 m)_0\tilde{\otimes}(h_2 n)_0) \otimes ((h_1 m)_1 (h_2 n)_1) \\
&= (h_1 m_0\tilde{\otimes}h_2 n_0) \otimes (m_1 n_1) \\
&= h(m_0\tilde{\otimes}n_0) \otimes (m_1 n_1) \\
&= h(m\tilde{\otimes}n)_0 \otimes (m \otimes n)_1.
\end{aligned}$$

The condition (2.1.9) is sitstified. Therefore $M\tilde{\otimes}_S N$ is an H-dimodule.
For all $h \in H, s \in S, m \in M$ and $n \in N$, we have

$$\begin{aligned}
h[s \rightharpoonup (m\tilde{\otimes}n)] &= h[(s \rightharpoonup m)\tilde{\otimes}n] \\
&= (h_1.(s \rightharpoonup m))\tilde{\otimes}(h_2.n) \\
&= ((h_1.s) \rightharpoonup (h_2.m))\tilde{\otimes}(h_3.n) \\
&= (h_1.s) \rightharpoonup [(h_2.m)\tilde{\otimes}(h_3.n)] \\
&= (h_1.s) \rightharpoonup [(h_{21}.m)\tilde{\otimes}(h_{22}.n)] \\
&= (h_1.s) \rightharpoonup (h_2.(m\tilde{\otimes}n)),
\end{aligned}$$

$$\begin{aligned}
[s \rightharpoonup (m\tilde{\otimes}n)]_0 \otimes [s \rightharpoonup (m\tilde{\otimes}n)]_1 &= [(s \rightharpoonup m)\tilde{\otimes}n]_0 \otimes [(s \rightharpoonup m)\tilde{\otimes}n]_1 \\
&= [(s \rightharpoonup m)_0\tilde{\otimes}n_0] \otimes (s \rightharpoonup m)_1 n_1 \\
&= [(s_0 \rightharpoonup m_0)\tilde{\otimes}n_0] \otimes (s_1 m_1)n_1 \\
&= [s_0 \rightharpoonup (m_0\tilde{\otimes}n_0)] \otimes s_1(m_1 n_1) \\
&= [s_0 \rightharpoonup (m\tilde{\otimes}n)_0] \otimes s_1(m\tilde{\otimes}n)_1.
\end{aligned}$$

So the relations (2.1.3) and (2.1.8) are sitisfied. Then $M\tilde{\otimes}_S N$ is an (S,H)-dimodule.

□

It is easy to see that for $M,N,P \in {}_S\mathscr{D}^H$, the natural map $(M\tilde{\otimes}_S N)\tilde{\otimes}_S P \to M\tilde{\otimes}_S(N\tilde{\otimes}_S P)$ is an (S,H)-dimodules isomorphism, and S is a unit with respect to $\tilde{\otimes}_S$. Therefore $({}_S\mathscr{D}^H, \tilde{\otimes}_S, S)$ is a monoidal category.

In the remainder of the paper, if M and N are (S,H)-dimodules, we denote by $Hom_S(M,N)$ the R-module of right S-linear maps from M to N considered as right S-modules and ${}_SHom(M,N)$ the R-module of left S-linear maps from M to N considered as left S-modules.

Lemma 2.2.2. *Let M and N be (S,H)-dimodules. Then the following hold:*

(i) *$Hom_S(M,N)$ is a left $S\#H$-module, where the action of S is defined by*

$$(s \rightharpoonup f)(m) = s \rightharpoonup f(m), \text{ for all } s \in S, f \in Hom_S(M,N), m \in M, \tag{2.2.10}$$

and the action of H is defined by

$$(hf)(m) = h_1[f(S_H(h_2)m)], \text{ for all } f \in Hom_S(M,N), h \in H, m \in M. \tag{2.2.11}$$

(ii) *If M is finitely generated projective as a right S-module, then $Hom_S(M,N)$ is an (S,H)-dimodule, where the coaction of H is defined by*

$$f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1 S_H(m_1), \text{ for all } f \in Hom_S(M,N), m \in M. \tag{2.2.12}$$

Proof.

(i) It is clear that the left S-action is well defined and $Hom_S(M,N)$ is a left S-module. Let $h \in H, m \in M, s \in S$, and $f \in Hom_S(M,N)$, we have:

$$\begin{aligned}
(hf)(m \leftharpoonup s) &= h_1[f(S_H(h_2)(m \leftharpoonup s)] \\
&= h_1[f((S_H(h_2)_1 m) \leftharpoonup (S_H(h_2)_2.s))] \\
&= h_1[f((S_H(h_{22})m) \leftharpoonup (S_H(h_{21}).s))] \\
&= h_1[(f(S_H(h_3)m)) \leftharpoonup (S_H(h_2).s)] \\
&= [h_{11}(f(S_H(h_3)m))] \leftharpoonup [h_{12}(S_H(h_2).s)] \\
&= [h_1(f(S_H(h_3)m))] \leftharpoonup [(h_{21}S_H(h_{22})).s] \\
&= ((hf)(m)) \leftharpoonup s
\end{aligned}$$

So $(hf) \in Hom_S(M,N)$, that is, the left H-action is well defined. It is easy to see that $Hom_S(M,N)$ is a left H-module. Let f be an element of $Hom_S(M,N)$. For all $h \in H, m \in M$, and $s \in S$, we have:

$$\begin{aligned}[h(s \rightharpoonup f)](m) &= h_1[(s \rightharpoonup f)(S_H(h_2)m)] \\ &= h_1[s \rightharpoonup (f(S_H(h_2)m)] \\ &= (h_1.s) \rightharpoonup [h_{21}(f(S_H(h_{22})m)] \\ &= [(h_1.s) \rightharpoonup (h_2 f)](m).\end{aligned}$$

So the relation (2.1.3) is satisfied. Therefore $Hom_S(M,N)$ is a left $S\#H$-module.

(ii) Let us consider the map

$$\rho : Hom_S(M,N) \to Hom_S(M,N \otimes H)$$

defined by

$$\rho(f)(m) = f(m_0)_0 \otimes f(m_0)_1 S_H(m_1), \text{ for all } f \in Hom_S(M,N), m \in M.$$

Since M is finitely generated projective as a right S-module,

$$Hom_S(M,N \otimes H) \cong Hom_S(M,N) \otimes H.$$

We put $\rho(f) = f_0 \otimes f_1$ if and only if

$$f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1 S_H(m_1).$$

So $Hom_S(M,N)$ becomes a right H-comodule. Let $f \in Hom_S(M,N), m \in M$ and $s \in S$, we have:

$$\begin{aligned}(s \rightharpoonup f)_0(m) \otimes (s \rightharpoonup f)_1 &= (s \rightharpoonup f)(m_0)_0 \otimes (s \rightharpoonup f)(m_0)_1 S_H(m_1) \\ &= (s \rightharpoonup f(m_0))_0 \otimes (s \rightharpoonup f(m_0))_1 S_H(m_1) \\ &= (s_0 \rightharpoonup f(m_0)_0) \otimes s_1 f(m_0)_1 S_H(m_1) \\ &= (s_0 \rightharpoonup f_0(m)) \otimes s_1 f_1 \\ &= (s_0 \rightharpoonup f_0)(m) \otimes s_1 f_1;\end{aligned}$$

so the condition (2.1.8) is satisfied.

For all $h \in H, m \in M,$ and $f \in Hom_S(M,N)$, we have:

$$\begin{aligned}(hf)_0(m) \otimes (hf)_1 &= [(hf)(m_0)]_0 \otimes [(hf)(m_0)]_1 S_H(m_1) \\ &= [h_1(f(S_H(h_2)m_0))]_0 \otimes [h_1(f(S_H(h_2)m_0))]_1 S_H(m_1) \\ &= h_1(f(S_H(h_2)m_0))_0 \otimes (f(S_H(h_2)m_0))_1 S_H(m_1) \\ &= h_1(f([S_H(h_2)m]_0)_0) \otimes (f([S_H(h_2)m]_0)_1) S_H([S_H(h_2)m)]_1) \\ &= h_1(f_0([S_H(h_2)m])) \otimes f_1 \\ &= (hf_0)(m) \otimes f_1,\end{aligned}$$

hence the condition (2.1.9) is satisfied. Therefore $Hom_S(M,N)$ is an (S,H)-dimodule.

□

Since S is not necessarily commutative, we need to consider the left and right S-module homomorphisms separately.

Lemma 2.2.3. *Let M and N be (S,H)-dimodules.*

(i) Then ${}_SHom(M,N)$ is a left $S\#H$-module, where the action of S is defined by

$$(s \rightharpoonup f)(m) = f(m \leftharpoonup s), \text{ for all } s \in S, f \in {}_SHom(M,N), m \in M, \tag{2.2.13}$$

and the coaction of H is defined by

$$(hf)(m) = h_1[f(S_H(h_2)m)], \text{ for all } f \in {}_SHom(M,N), h \in H, m \in M. \tag{2.2.14}$$

(ii) If M is finitely generated projective as a left S-module, then ${}_SHom(M,N)$ is an (S,H)-dimodule, where the coaction of H is defined by

$$f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1 S_H(m_1), \text{ for all } f \in {}_SHom(M,N), m \in M. \tag{2.2.15}$$

Proof.

(i) Let $f \in {}_SHom(M,N)$, $h \in H$, $s \in S$ and $n \in N$. We have

$$\begin{aligned}
(hf)(s \rightharpoonup m) &= h_1[f(S_H(h_2)(s \rightharpoonup m))] \\
&= h_1[f(S_H(h_2)_1.s \rightharpoonup S_H(h_2)_2 m)] \\
&= h_1[f(S_H(h_{22}).s \rightharpoonup S_H(h_{21})m)] \\
&= h_1[S_H(h_{22}).s \rightharpoonup f(S_H(h_{21})m))] \\
&= h_1[S_H(h_3).s \rightharpoonup f(S_H(h_2)m))] \\
&= h_1[S_H(h_2).s \rightharpoonup f(S_H(h_3)m))] \\
&= [h_{11}.(S_H(h_2).s)] \rightharpoonup [h_{12}.(f(S_H(h_3)m)))] \\
&= [(h_{11}S_H(h_2)).s] \rightharpoonup [h_{12}.(f(S_H(h_3)m))] \\
&= [(h_1S_H(h_3)).s] \rightharpoonup [h_2.(f(S_H(h_4)m))] \\
&= [(h_1S_H(h_2)).s] \rightharpoonup [h_3.(f(S_H(h_4)m))] \\
&= [\varepsilon(h_1)s] \rightharpoonup [h_2(f(S_H(h_3)m))] \\
&= s \rightharpoonup [\varepsilon(h_1)h_2(f(S_H(h_3)m))] \\
&= s \rightharpoonup [h_1(f(S_H(h_2)m))] \\
&= s \rightharpoonup ((hf)(m))
\end{aligned}$$

So $hf \in {}_SHom(M,N)$, that is, the H-action is well defined. It is easy to see that ${}_SHom(M,N)$ is a left H-module. For $s' \in S$, $m \in M$ and $f \in {}_SHom(M,N)$ we have

$$\begin{aligned}(s \rightharpoonup f)(s' \rightharpoonup m) &= f[(s' \rightharpoonup m) \leftharpoonup s]\\ &= f[s' \rightharpoonup (m \leftharpoonup s)]\\ &= s' \rightharpoonup (f(m \leftharpoonup s))\\ &= s' \rightharpoonup ((s \rightharpoonup f)(m))\end{aligned}$$

So $(s \rightharpoonup f) \in {}_SHom(M,N)$, that is, the left S-action is well defined. It is easy to see that ${}_SHom(M,N)$ is a left S-module.
We have for all $f \in {}_SHom(M,N), s \in S, h \in H$, and $m \in M$,

$$\begin{aligned}[(h_1.s) \rightharpoonup (h_2f)](m) &= (h_2f)(m \leftharpoonup (h_1.s))\\ &= (h_1f)(m \leftharpoonup (h_2.s))\\ &= h_{11}(f[S_H(h_{12})(m \leftharpoonup (h_2.s)])\\ &= h_1(f[S_H(h_2)(m \leftharpoonup (h_3.s)])\\ &= h_1(f[(S_H(h_2)_1m) \leftharpoonup (S_H(h_2)_2(h_3.s))])\\ &= h_1(f[(S_H(h_{22})m) \leftharpoonup ((S_H(h_{21})h_3).s)])\\ &= h_1(f[(S_H(h_3)m) \leftharpoonup ((S_H(h_2)h_4).s)])\\ &= h_1(f[(S_H(h_2)m) \leftharpoonup ((S_H(h_3)h_4).s)])\\ &= h_1(f[(S_H(h_2)m) \leftharpoonup ((\varepsilon(h_3)1_H).s)])\\ &= h_1(f[(S_H(h_2)m) \leftharpoonup s])\\ &= h_1[(s \rightharpoonup f)(S_H(h_2)m)]\\ &= [h(s \rightharpoonup f)](m)\end{aligned}$$

and (14) is satisfied. Therefore ${}_SHom(M,N)$ is a left $S\#H$-module.

(ii) When M is a finitely generated projective left S-module,

$${}_SHom(M,N) \otimes H \cong {}_SHom(M, N \otimes H),$$

so ${}_SHom(M,N)$ becomes an H-comodule with the given coaction. We defined $\rho(f)$ as in the proof of Lemma 2.2.2. Indeed: for all $f \in {}_SHom(M,N), s \in S$, and $m \in M$, we have

$$\begin{aligned}f_0(s \rightharpoonup m) \otimes f_1 &= f((s \rightharpoonup m)_0)_0 \otimes f((s \rightharpoonup m)_1)_1 S_H((s \rightharpoonup m)_1)\\ &= f(s_0 \rightharpoonup m_0)_0 \otimes f(s_0 \rightharpoonup m_0)_1 S_H(s_1m_1)\\ &= (s_0 \rightharpoonup f(m_0))_0 \otimes (s_0 \rightharpoonup f(m_0))_1 S_H(s_1m_1)\\ &= (s_{00} \rightharpoonup f(m_0)_0) \otimes s_{01} f(m_0)_1 S_H(m_1) S_H(s_1)\\ &= (s_{00} \rightharpoonup f(m_0)_0) \otimes f(m_0)_1 S_H(m_1) s_{01} S_H(s_1)\\ &= (s_0 \rightharpoonup f(m_0)_0) \otimes f(m_0)_1 S_H(m_1)\varepsilon(s_1)\\ &= (s \rightharpoonup f(m_0)_0) \otimes f(m_0)_1 S_H(m_1)\\ &= (s \rightharpoonup (f_0(m))) \otimes f_1\end{aligned}$$

So $f_0 \in {}_SHom(M,N)$, that is, the right H-coaction is well defined. It is easy to see that ${}_SHom(M,N)$ is a right H-comodule. We have

$$\begin{aligned}
[(s_0 \rightharpoonup f_0)(m)] \otimes s_1 f_1 &= [f_0(m \leftharpoonup s_0)] \otimes s_1 f_1 \\
&= [f((m \leftharpoonup s_0)_0)]_0 \otimes s_1 [f((m \leftharpoonup s_0)_0)]_1 S_H((m \leftharpoonup s_0)_1) \\
&= [f((m \leftharpoonup s_0)_0)]_0 \otimes [f((m \leftharpoonup s_0)_0)]_1 S_H((m \leftharpoonup s_0)_1) s_1 \\
&= [f(m_0 \leftharpoonup s_{00})]_0 \otimes [f(m_0 \leftharpoonup s_{00})]_1 S_H(m_1 s_{01}) s_1 \\
&= [f(m_0 \leftharpoonup s_0)]_0 \otimes [f(m_0 \leftharpoonup s_0)]_1 S_H(s_{11}) S_H(m_1) s_{12} \\
&= [f(m_0 \leftharpoonup s_0)]_0 \otimes [f(m_0 \leftharpoonup s_0)]_1 S_H(m_1) S_H(s_{11}) s_{12} \\
&= [f(m_0 \leftharpoonup s_0)]_0 \otimes [f(m_0 \leftharpoonup s_0)]_1 S_H(m_1) \varepsilon(s_1) \\
&= [f(m_0 \leftharpoonup s)]_0 \otimes [f(m_0 \leftharpoonup s)]_1 S_H(m_1) \\
&= [(s \rightharpoonup f)(m_0)]_0 \otimes [(s \rightharpoonup f)(m_0)]_1 S_H(m_1) \\
&= (s \rightharpoonup f)_0(m) \otimes (s \rightharpoonup f)_1
\end{aligned}$$

for all $f \in {}_SHom(M,N), s \in S$, and $m \in M$, and so the equation (2.1.8) is satisfied.Therefore ${}_SHom(M,N)$ is a right H-comodule. For all $f \in {}_SHom(M,N), h \in H$, and $m \in M$, we have

$$(hf)_0(m) \otimes (hf)_1 = (hf_0)(m) \otimes f_1, \text{ see Lemma 2.2.2 above.}$$

□

Lemma 2.2.4. *Let N,P and Q be (S,H)-dimodules.*

(i) *If P is finitely generated projective as a right S-module, then we have an R-module isomorphism*

$$_{S\#H}Hom^H(N \tilde{\otimes}_S P, Q) \cong {}_{S\#H}Hom^H(N, Hom_S(P,Q));$$

(ii) *if P is finitely generated projective as a left S-module, then we have an R-module isomorphism*

$$_{S\#H}Hom^H(P \tilde{\otimes}_S N, Q) \cong {}_{S\#H}Hom^H(N, {}_SHom(P,Q)).$$

Proof.

(i) We consider the R-linear map

$$\phi : {}_{S\#H}Hom^H(N \tilde{\otimes}_S P, Q) \to {}_{S\#H}Hom^H(N, Hom_S(P,Q))$$

given by $\phi(f)(n)(p) = f(n \tilde{\otimes} p)$. Let f be an element of ${}_{S\#H}Hom^H(N \tilde{\otimes}_S P, Q)$. For all $n \in N, p \in P$, and $s \in S$, we have

$$\begin{aligned}
\phi(f)(n)(p \leftharpoonup s) &= f(n \tilde{\otimes} (p \leftharpoonup s)) \\
&= f((n \tilde{\otimes} p) \leftharpoonup s) \\
&= f((n \tilde{\otimes} p)_1.s \rightharpoonup (n \tilde{\otimes} p)_0) \\
&= ((n \tilde{\otimes} p)_1.s) \rightharpoonup f((n \tilde{\otimes} p)_0) \\
&= (f(n \tilde{\otimes} p)_1.s) \rightharpoonup f(n \tilde{\otimes} p)_0 \\
&= f(n \tilde{\otimes} p) \leftharpoonup s \\
&= [\phi(f)(n)(p)] \leftharpoonup s
\end{aligned}$$

So $\phi(f)(n)(p)$ is right S-linear. We also have:

$$[\phi(f)(s \rightharpoonup n)](p) = f[(s \rightharpoonup n)\tilde{\otimes} p] = f[s \rightharpoonup (n\tilde{\otimes} p)] = s \rightharpoonup [f(n\tilde{\otimes} p)] = s \rightharpoonup [\phi(f)(n)](p).$$

So $\phi(f)$ is left S-linear. Let $h \in H$,we have :

$$\begin{aligned}
\phi(f)(hn)(p) &= f[(hn)\tilde{\otimes} p] \\
&= f[(h_1\varepsilon(h_2)n)\tilde{\otimes} p] \\
&= f[(h_1 n)\tilde{\otimes}\varepsilon(h_2)p] \\
&= f[(h_1 n)\tilde{\otimes}\varepsilon(h_2)1_H p] \\
&= f[(h_1 n)\tilde{\otimes} h_{21}S_H(h_{22})p] \\
&= f[(h_{11} n)\tilde{\otimes} h_{12}S_H(h_2)p] \\
&= f[h_1(n\tilde{\otimes} S_H(h_2)p)] \\
&= h_1[f(n\tilde{\otimes} S_H(h_2)p)] \\
&= h_1[\phi(f)(n)(S_H(h_2)p)] \\
&= [h(\phi(f)(n))](p),
\end{aligned}$$

So, $\phi(f)$ is left H-linear.Therefore, $\phi(f)$ is $S\#H$-linear.

Since f is a right H-linear map, we have:

$$\begin{aligned}
[\phi(f)(n)]_0(p)\otimes[\phi(f)(n)]_1 &= [\phi(f)(n)(p_0)]_0\otimes[\phi(f)(n)(p_0)]_1 S_H(p_1) \\
&= [f(n\tilde{\otimes} p_0)]_0\otimes[f(n\tilde{\otimes} p_0)]_1 S_H(p_1) \\
&= f((n\tilde{\otimes} p_0)_0)\otimes(n\tilde{\otimes} p_0)_1 S_H(p_1) \\
&= f(n_0\tilde{\otimes} p_{00})\otimes n_1 p_{01} S_H(p_1) \\
&= f(n_0\tilde{\otimes} p_0)\otimes n_1 p_{11} S_H(p_{12}) \\
&= f(n_0\tilde{\otimes} p_0)\otimes n_1\varepsilon(p_1) \\
&= f(n_0\tilde{\otimes} p_0\varepsilon(p_1))\otimes n_1 \\
&= f(n_0\tilde{\otimes} p)\otimes n_1 \\
&= \phi(f)(n_0)(p)\otimes n_1.
\end{aligned}$$

We deduce that $[\phi(f)(n)]_0\otimes[\phi(f)(n)]_1 = [\phi(f)(n_0)]\otimes n_1$, that is, $\phi(f)$ is right H-colinear.
It follows taht ϕ is well defined.
Let us consider the R-linear map

$$\psi : {}_{S\#H}Hom^H(N, Hom_S(P,Q)) \to {}_{S\#H}Hom^H(N\tilde{\otimes}_S P, Q)$$

defined by $\psi(g)(n\tilde{\otimes} p) = g(n)(p)$, for all $g \in {}_{S\#H}Hom^H(N, Hom_S(P,Q)), n \in N$, and $p \in P$.
for $h \in H$, we have:

$$\begin{aligned}
\psi(g)(h(n\tilde{\otimes}p)) &= \psi(g)(h_1n\tilde{\otimes}h_2p)\\
&= g(h_1n)(h_2p)\\
&= [h_1(g(n))](h_2p)\\
&= h_{11}[g(n)(S_H(h_{12})h_2p)]\\
&= h_1[g(n)(S_H(h_{21})h_{22}p)]\\
&= h_1[g(n)(\varepsilon(h_2)p)]\\
&= h[g(n)(p)]\\
&= h(\psi(g)(n\tilde{\otimes}p),
\end{aligned}$$

so $\psi(g)$ is left H-linear. Let $s \in S$, we have:

$$\begin{aligned}
\psi(g)[s \rightharpoonup (n\tilde{\otimes}p)] &= \psi(g)[(s \rightharpoonup n)\tilde{\otimes}p]\\
&= g(s \rightharpoonup n)(p)\\
&= [s \rightharpoonup (g(n))](p)\\
&= s \rightharpoonup [g(n)(p)]\\
&= s \rightharpoonup [\psi(g)(n\tilde{\otimes}p)],
\end{aligned}$$

so $\psi(g)$ is left S-linear. Therefore $\psi(g)$ is $S\#H$-linear. Since g is a right H-colinear map, we have:

$$\begin{aligned}
[\psi(g)(n\tilde{\otimes}p)]_0 \otimes [\psi(g)(n\tilde{\otimes}p)]_1 &= [g(n)(p)]_0 \otimes [g(n)(p)]_1\\
&= [g(n)(p_0)]_0 \otimes [g(n)(p_0)]_1\varepsilon(p_1)\\
&= [g(n)(p_0)]_0 \otimes [g(n)(p_0)]_1 S_H(p_{11})p_{12}\\
&= [g(n)(p_{00})]_0 \otimes [g(n)(p_{00})]_1 S_H(p_{01})p_1\\
&= [g(n)_0(p_0)] \otimes g(n)_1 p_1\\
&= [g(n_0)(p_0)] \otimes n_1p_1\\
&= [\psi(g)(n_0\tilde{\otimes}p_0)] \otimes n_1p_1\\
&= [\psi(g)((n\tilde{\otimes}p)_0)] \otimes (n\tilde{\otimes}p)_1.
\end{aligned}$$

We deduce that $\psi(g)$ is right H-colinear. It follows that ψ is well defined.

Finally we have, for all $f \in {}_{S\#H}Hom^H(N\tilde{\otimes}_S P, Q)$, $g \in {}_{S\#H}Hom^H(N, Hom_S(P,Q))$, $n \in N$, and $p \in P$,

$$[(\phi\circ\psi)(g)](n)(p) = \psi(g)(n\tilde{\otimes}p) = g(n)(p)$$

and

$$[(\psi\circ\phi)(f)](n\tilde{\otimes}p) = \phi(f)(n)(p) = f(n\tilde{\otimes}p).$$

So ϕ and ψ are inverse of each other.

(ii) Let f be an element of ${}_{S\#H}Hom^H(P\tilde{\otimes}_S N,Q)$. We consider the R-linear map

$$\phi : {}_{S\#H}Hom^H(P\tilde{\otimes}_S N,Q) \longrightarrow {}_{S\#H}Hom^H(N,{}_SHom(P,Q))$$

defined by $\phi(f)(n)(p) = f(p\tilde{\otimes}n)$. We have for all $n \in N, p \in P$, and $s \in S$,

$$\phi(f)(n)(s \rightharpoonup p) = f((s \rightharpoonup p)\tilde{\otimes}n) = f(s \rightharpoonup (p\tilde{\otimes}n)) = s \rightharpoonup (f(p\tilde{\otimes}n)) = s \rightharpoonup [\phi(f)(n)(p)]$$

So $\phi(f)(n)$ is S-linear. For all $f \in {}_{S\#H}Hom^H(N\tilde{\otimes}_S P,Q), n \in N, p \in P$, and $s \in S$, we have:

$$\begin{aligned}
[\phi(f)(s \rightharpoonup n)](p) &= f(p\tilde{\otimes}(s \rightharpoonup n)) \\
&= f(p\tilde{\otimes}(n_0 \leftharpoonup S_H(n_1).s)) \\
&= f((p\tilde{\otimes}n_0) \leftharpoonup (S_H(n_1).s)) \\
&= f[((p\tilde{\otimes}n_0)_1 S_H(n_1).s) \rightharpoonup (p\tilde{\otimes}n_0)_0] \\
&= f[((p_1 n_{01})S_H(n_1).s) \rightharpoonup (p_0\tilde{\otimes}n_{00})] \\
&= f[(p_1 n_{01}S_H(n_1)).s) \rightharpoonup (p_0\tilde{\otimes}n_{00})] \\
&= f[((p_1 n_{11}S_H(n_{12})).s) \rightharpoonup (p_0\tilde{\otimes}n_0)] \\
&= f[((p_1\varepsilon(n_1)).s) \rightharpoonup (p_0\tilde{\otimes}n_0)] \\
&= f[(p_1.s) \rightharpoonup (p_0\tilde{\otimes}n_0\varepsilon(n_1))] \\
&= f[((p_1.s) \rightharpoonup p_0)\tilde{\otimes}n] \\
&= f[((p \leftharpoonup s)\tilde{\otimes}n] \\
&= \phi(f)(n)(p \leftharpoonup s) \\
&= [s \rightharpoonup \phi(f)(n)](p).
\end{aligned}$$

So $\phi(f)$ is left S-linear. Let $h \in H$, we have

$$\begin{aligned}
\phi(f)(hn)(p) &= f[p\tilde{\otimes}(hn)] \\
&= f[\varepsilon(h_1)p\tilde{\otimes}h_2 n] \\
&= f[h_{11}S_H(h_{12})p\tilde{\otimes}h_2 n] \\
&= f[h_{11}S_H(h_2)p\tilde{\otimes}h_{12} n] \\
&= f[h_1(S_H(h_2)p\tilde{\otimes}n)] \\
&= h_1[f(S_H(h_2)p\tilde{\otimes}n)] \\
&= h_1[\phi(f)(n)(S_H(h_2)p)] \\
&= [h(\phi(f)(n))](p).
\end{aligned}$$

So $\phi(f)$ is left H-linear.

Therefore $\phi(f)$ is $S\#H$-linear. Since f is left H-colinear, we have

$$\begin{aligned}
[\phi(f)(n)]_0(p) \otimes [\phi(f)(n)]_1 &= [(\phi(f)(n))(p_0)]_0 \otimes [(\phi(f)(n))(p_0)]_1 S_H(p_1) \\
&= [f(p_0\tilde{\otimes}n]_0 \otimes [f(p_0\tilde{\otimes}n]_1 S_H(p_1) \\
&= f[(p_0\tilde{\otimes}n)_0] \otimes (p_0\tilde{\otimes}n)_1 S_H(p_1) \\
&= f(p_{00}\tilde{\otimes}n_0) \otimes n_1 p_{01} S_H(p_1) \\
&= f(p_0\tilde{\otimes}n_0) \otimes n_1\varepsilon(p_1) \\
&= f(p\tilde{\otimes}n_0) \otimes n_1 \\
&= \phi(f)(n_0)(p) \otimes n_1.
\end{aligned}$$

We deduce that $[\phi(f)(n)]_0 \otimes [\phi(f)(n)]_1 = \phi(f)(n_0)\otimes n_1$, that is, $\phi(f)$ is right H-colinear. It follows that ϕ is well defined.
Let g be an element of ${}_{S\#H}Hom^H(N, {}_SHom(P,Q))$. We consider now the R-linear map

$$\psi : {}_{S\#H}Hom^H(N, {}_SHom(P,Q)) \longrightarrow {}_{S\#H}Hom^H(P\tilde{\otimes}_S N, Q),$$

defined by

$$\psi(g)(p\tilde{\otimes} n) = g(n)(p), \text{ for all } p \in P, n \in N.$$

We have;

$$\begin{aligned}
\psi(g)[h(p\tilde{\otimes} n)] &= \psi(g)[h_1 p \tilde{\otimes} h_2 n)] \\
&= g(h_2 n)(h_1 p) \\
&= g(h_1 n)(h_2 p) \\
&= h_{11}[(g(n))(S_H(h_{12})h_2 p)] \\
&= h_1[(g(n))(S_H(h_{21})h_{22} p)] \\
&= h_1[g(n)(\varepsilon(h_2)p)] \\
&= h[g(n)(p)] \ = \ h[\psi(g)(p\tilde{\otimes} n)],
\end{aligned}$$

so $\psi(g)$ is left H-linear. Let $s \in S$, we have

$$\psi(g)[s \rightharpoonup (p\tilde{\otimes} n)] = \psi(g)[(s \rightharpoonup p)\tilde{\otimes} n] = g(n)(s \rightharpoonup p) = s \rightharpoonup [g(n)(p)] = s \rightharpoonup [\psi(g)(p\tilde{\otimes} n)]$$

so $\psi(g)$ is left S-linear.

Since g is right H-colinear. For all $p \in P, n \in N$, we have:

$$\begin{aligned}
[\psi(g)(p\tilde{\otimes} n)]_0 \otimes [\psi(g)(p\tilde{\otimes} n)]_1 &= [g(n)(p)]_0 \otimes [g(n)(p)]_1 \\
&= [g(n)(p_0)]_0 \otimes [g(n)(p_0)]_1 \varepsilon(p_1) \\
&= [g(n)(p_0)]_0 \otimes [g(n)(p_0)]_1 S_H(p_1) p_2 \\
&= [g(n)(p_{00})]_0 \otimes [g(n)(p_{00})]_1 S_H(p_{01}) p_1 \\
&= [g(n)_0(p_0)] \otimes [g(n)_1] p_1 \\
&= [g(n_0)(p_0)] \otimes (n_1 p_1) \\
&= [\psi(g)(p_0 \tilde{\otimes} n_0)] \otimes (n_1 p_1) \\
&= [\psi(g)((p\tilde{\otimes} n)_0)] \otimes (p\tilde{\otimes} n)_1.
\end{aligned}$$

We deduce that $\psi(g)$ is right H-colinear. It follows that ψ is well-defined. It is easy to see that ϕ and ψ are inverse of each other. We have

$$[(\phi \circ \psi)(g)](n)(p) = \psi(g)(p\tilde{\otimes} n) = g(n)(p)$$

and

$$[(\psi \circ \phi)(f)](p\tilde{\otimes} n) = \phi(f)(n)(p) = f(p\tilde{\otimes} n),$$

for all $n \in N, p \in P, f \in {}_{S\#H}Hom^H(N\tilde{\otimes}_S P, Q)$, and $g \in {}_{S\#H}Hom^H(N, {}_SHom(P,Q))$.

The maps ϕ and ψ are inverse of each other.

□

From Lemma 2.2.4 (i), we deduce that the functor $Hom_S(P,-)$ defined from ${}_S\mathscr{D}^H$ to ${}_S\mathscr{D}^H$ with P finitely generated projective as a right S-module is right adjoint to the functor $-\tilde{\otimes}_S P$ defined from ${}_S\mathscr{D}^H$ to ${}_S\mathscr{D}^H$. In the notation of [59], $Hom_S(P,Q) = [P,Q]$. It also follows from Lemma 2.2.4 (i), that if N and P are projective as right S-modules, then $N\tilde{\otimes}_S P$ is projective as a right S-module.

From Lemma 2.2.4 (ii), we deduce that the functor ${}_SHom(P,-)$ defined from ${}_S\mathscr{D}^H$ to ${}_S\mathscr{D}^H$ with P finitely generated projective as a left S-module is right adjoint to the functor $P\tilde{\otimes}_S-$ defined from ${}_S\mathscr{D}^H$ to ${}_S\mathscr{D}^H$. In the notation of [59], ${}_SHom(P,Q) = \{P,Q\}$. It also follows from Lemma 2.2.4 (ii) that if N and P are projective as left S-modules, then $P\tilde{\otimes}_S N$ is projective as a left S-module.

The results of the following lemma are useful for some computations.

Lemma 2.2.5. *Let M,N be (S,H)-dimodules.*

(i) *If M is finitely generated projective as a right S-module, then*

$$(f \leftharpoonup s)(m) = f(s \rightharpoonup m), \tag{2.2.16}$$

for all $f \in Hom_S(M,N), m \in M$, and $s \in S$.

(ii) *If M is finitely generated projective as a left S-module, then*

$$(f \leftharpoonup s)(m) = f(m) \leftharpoonup s, \tag{2.2.17}$$

for all $f \in {}_SHom(M,N), m \in M$, and $s \in S$.

Proof.

(i) For all $f \in Hom_S(M,N), m \in M$, and $s \in S$, we have:

$$\begin{aligned} f(s \rightharpoonup m) &= f(m_0 \leftharpoonup S_H(m_1).s) \\ &= f(m_0) \leftharpoonup S_H(m_1).s \\ &= (f(m_0)_1 S_H(m_1).s) \rightharpoonup (f(m_0)_0) \\ &= (f_1.s) \rightharpoonup (f_0(m)) \\ &= (f_1.s \rightharpoonup f_0)(m) \\ &= (f \leftharpoonup s)(m). \end{aligned}$$

(ii) For all $f \in {}_SHom(M,N), m \in M$, and $s \in S$, we have:

$$\begin{aligned}
(f \leftharpoonup s)(m) &= (f_1.s \rightharpoonup f_0)(m)\\
&= (f_1.s) \rightharpoonup f_0(m)\\
&= [(f(m_0)_1 S_H(m)_1).s] \rightharpoonup f(m_0)_0\\
&= f(m_0)_{00} \leftharpoonup S_H(f(m_0)_{01})[(f(m_0)_1 S_H(m)_1).s]\\
&= f_{00}(m) \leftharpoonup S_H(f_{01})(f_1.s)\\
&= f_0(m) \leftharpoonup S_H(f_{11})(f_{12}.s)\\
&= f_0(m) \leftharpoonup (S_H(f_{11})f_{12}).s\\
&= f_0(m) \leftharpoonup \varepsilon(f_1)s\\
&= (f_0\varepsilon(f_1))(m) \leftharpoonup s\\
&= f(m) \leftharpoonup s.
\end{aligned}$$

□

From Sections 3 to 6, H is a commutative and cocommutative Hopf algebra and S is an H-commutative H-dimodule algebra.

2.3 The category of dyslectic (S,H)-dimodules

Our objective for this section is to define the subcategory of dyslectic (S,H)-dimodules. Note that an (S,H)-dimodule is just a left S-module in the braided monoidal category $\mathscr{D}^H$.

An object M of ${}_S\mathscr{D}^H$ is *dyslectic* if $h_M \circ \gamma_{M,S} \circ \gamma_{S,M} = h_M$, where $h_M : S \otimes M \to M$ denotes the left action of S on M [50] . It follows that an object M of ${}_S\mathscr{D}^H$ is dyslectic if and only if

$$s \rightharpoonup m = ((s_1.m)_1 s_0) \rightharpoonup (s_1 m)_0. \tag{2.3.1}$$

In our context, that means

$$s \rightharpoonup m = (m_1.s_0) \rightharpoonup (s_1 m_0). \tag{2.3.2}$$

Clearly, S is a *dyslectic* (S,H)*-dimodule*, and every H-dimodule can be regarded as a dyslectic (R,H)-dimodule. A *dyslectic* (S,H)*-dimodule homomorphism* is an (S,H)-dimodule homomorphism between dyslectic (S,H)-dimodules. Let M be an (S,H)-dimodule and let us consider the condition

$$s \rightharpoonup m = (s_1 m) \leftharpoonup s_0, \tag{2.3.3}$$

which is equivalent to the equation

$$m \leftharpoonup s = s_0 \rightharpoonup (S_H(s_1)m). \tag{2.3.4}$$

Lemma 2.3.1. (i) *Let N be an (S,H)-dimodule. Then the condition (2.3.3) is satisfied for N if and only if the braiding map $\gamma_{M,N} : M\tilde{\otimes}N \to N\tilde{\otimes}M$ induces a well-defined map denoted again $\gamma_{M,N} : M\tilde{\otimes}_S N \to N\tilde{\otimes}_S M$, defined by*

$$\gamma_{M,N}(m\tilde{\otimes}_S n) = m_1 n\tilde{\otimes}_S m_0,$$

for all M in ${}_S\mathscr{D}^H$.

(ii) *Let M be an (S,H)-dimodule. Then the condition (2.3.4) is satisfied for M if and only if the braiding map $\gamma^{-1}_{M,N}: M\tilde{\otimes}N \to N\tilde{\otimes}M$ induces a well-defined map denoted again $\gamma^{-1}_{M,N}$: $M\tilde{\otimes}_S N \to N\tilde{\otimes}_S M$, defined by*

$$\gamma^{-1}_{M,N}(m\tilde{\otimes}_S n) = n_0\tilde{\otimes}_S S_H(n_1)m,$$

for all N in ${}_S\mathscr{D}^H$.

Proof.

(i) Let $m \in M, n \in N$ and $s \in S$. If (2.3.3) is satisfied for N, then

$$\begin{aligned}
\gamma_{M,N}((m \leftharpoonup s)\tilde{\otimes}n) &= \gamma_{M,N}((m_1 s \rightharpoonup m_0)\tilde{\otimes}n)\\
&= (m_1 s \rightharpoonup m_0)_1 n\tilde{\otimes}(m_1 s \rightharpoonup m_0)_0\\
&= ((m_1 s)_1 m_{01} n)\tilde{\otimes}((m_1 s)_0 \rightharpoonup m_{00})\\
&= ((s_1 m_{01})n)\tilde{\otimes}(m_1 s_0 \rightharpoonup m_{00})\\
&= [((s_1 m_{01})n) \leftharpoonup (m_1.s)]\tilde{\otimes}m_{00}\\
&= [((s_1 m_{11})n) \leftharpoonup (m_{12}.s)]\tilde{\otimes}m_0\\
&= [m_1((s_1 n) \leftharpoonup s_0)]\tilde{\otimes}m_0\\
&\overset{(2.3.3)}{=} (m_1(s \rightharpoonup n))\tilde{\otimes}m_0\\
&= \gamma_{M,N}(m\tilde{\otimes}(s \rightharpoonup n)).
\end{aligned}$$

So $\gamma_{M,N}$ is well defined. If $\gamma_{M,N}$ is well defined for all M in ${}_S\mathscr{D}^H$, then $\gamma_{S,N}$ is well defined. Let $n \in N$ and $s \in S$. We have

$$\gamma_{S,N}((1_S \leftharpoonup s)\tilde{\otimes}n) = \gamma_{S,N}(1_S\tilde{\otimes}(s \rightharpoonup n)).$$

We also have

$$\begin{aligned}
\gamma_{S,N}((1_S \leftharpoonup s)\tilde{\otimes}n) &= ((1_S \leftharpoonup s)_1 n)\tilde{\otimes}(1_S \leftharpoonup s)_0\\
&= (1_H s_1 n)\tilde{\otimes}(1_S \leftharpoonup s_0)\\
&= ((s_1 n)\tilde{\otimes}1_S) \leftharpoonup s_0\\
&= ((s_1 n) \leftharpoonup s_0)\tilde{\otimes}1_S
\end{aligned}$$

and $\quad \gamma_{S,N}(1_S\tilde{\otimes}(s \rightharpoonup n)) = 1_H(s \rightharpoonup n)\tilde{\otimes}1_S = (s \rightharpoonup n)\tilde{\otimes}1_S.$

We deduce that $\quad ((s_1 n) \leftharpoonup s_0)\tilde{\otimes}1_S = (s \rightharpoonup n)\tilde{\otimes}1_S$

hence $\quad s \rightharpoonup n = (s_1 n) \leftharpoonup s_0.\quad$ So (2.3.3) is satisfied for N.

(ii) For all $m \in M, n \in N$ and $s \in S$, we have:

$$\begin{aligned}
\gamma^{-1}_{M,N}((m \leftharpoonup s)\tilde{\otimes} n) &= n_0\tilde{\otimes}(S_H(n_1)(m \leftharpoonup s)) \\
&\overset{(2.3.4)}{=} n_0\tilde{\otimes}(S_H(n_1)(s_0 \rightharpoonup (S_H(s_1)m)), \\
&= n_0\tilde{\otimes}[(S_H(n_1)_1.s_0) \rightharpoonup (S_H(n_1)_2(S_H(s_1)m))] \\
&= n_0\tilde{\otimes}[(S_H(n_2).s_0) \rightharpoonup (S_H(n_1)(S_H(s_1)m))] \\
&= n_0\tilde{\otimes}[(S_H(n_1).s_0) \rightharpoonup (S_H(n_2)(S_H(s_1)m))] \\
&= [n_0 \leftharpoonup (S_H(n_1).s_0)]\tilde{\otimes}(S_H(n_2)(S_H(s_1)m)) \\
&= [n_{00} \leftharpoonup (S_H(n_{01}).s_0)]\tilde{\otimes}(S_H(n_1)(S_H(s_1)m)) \\
&= (s_0 \rightharpoonup n_0)\tilde{\otimes}(S_H(s_1n_1)m) \\
&= (s \rightharpoonup n)_0\tilde{\otimes}(S_H((s \rightharpoonup n)_1)m) \\
&= \gamma^{-1}_{M,N}(m\tilde{\otimes}(s \rightharpoonup n)).
\end{aligned}$$

So $\gamma^{-1}_{M,N}$ is well defined.
If $\gamma^{-1}_{M,N}$ is well defined, then $\gamma^{-1}_{M,S}$ is well defined. For all $m \in M$ and $s \in S$, we have:

$$\gamma^{-1}_{M,S}(m\tilde{\otimes}(s \rightharpoonup 1_S)) = \gamma^{-1}_{M,S}((m \leftharpoonup s)\tilde{\otimes}1_S).$$

We also have

$$\begin{aligned}
\gamma^{-1}_{M,S}(m\tilde{\otimes}(s \rightharpoonup 1_S)) &= (s \rightharpoonup 1_S)_0\tilde{\otimes}(S_H((s \rightharpoonup 1_S)_1)m) \\
&= (s_0 \rightharpoonup 1_S)\tilde{\otimes}(S_H(s_1 1_H)m) \\
&= s_0 \rightharpoonup (1_S\tilde{\otimes}(S_H(s_1)m)) \\
&= (1_S\tilde{\otimes}(S_H(s_1)m)_0) \leftharpoonup [S_H((1_S\tilde{\otimes}(S_H(s_1)m)_1).s_0] \\
&= (1_S\tilde{\otimes}(S_H(s_1)m_0)) \leftharpoonup [S_H(1_H m_1).s_0] \\
&= 1_S\tilde{\otimes}[(S_H(s_1)m_0) \leftharpoonup (S_H(m_1).s_0)] \\
&= 1_S\tilde{\otimes}[((S_H(s_1)m_0)_1(S_H(m_1).s_0)) \rightharpoonup (S_H(s_1)m_0)_0] \\
&= 1_S\tilde{\otimes}[(m_{01}(S_H(m_1).s_0)) \rightharpoonup (S_H(s_1)m_{00})] \\
&= 1_S\tilde{\otimes}[((m_{11}S_H(m_{12})).s_0) \rightharpoonup (S_H(s_1)m_0)] \\
&= 1_S\tilde{\otimes}[(\varepsilon(m_1)s_0) \rightharpoonup (S_H(s_1)m_0)] \\
&= 1_S\tilde{\otimes}[s_0 \rightharpoonup (S_H(s_1)m)]
\end{aligned}$$

and

$$\gamma^{-1}_{M,S}((m \leftharpoonup s)\tilde{\otimes}1_S) = 1_S\tilde{\otimes}(S_H(1_H)(m \leftharpoonup s)) = 1_S\tilde{\otimes}(m \leftharpoonup s).$$

So the condition (2.3.4) is satified for M.

□

The following lemma provides an easiest necessary and sufficient condition to show that an (S,H)-dimodule is dyslectic.

Lemma 2.3.2. *Let M be an (S,H)-dimodule. Then M is dyslectic if and only if the condition (2.3.3) is satisfied for M.*

Proof. Assume the condition (2.3.3) is satisfied for M. Then we have

$$s \rightharpoonup m = (s_1m) \leftharpoonup s_0 = ((s_1m)_1.s_0) \rightharpoonup (s_1m)_0 = (m_1.s_0) \rightharpoonup (s_1m_0),$$

and the condition (2.3.2) if satisfied for M, that is, M is dyslectic. Conversely, if M dyslectic, then

$$
\begin{aligned}
s \rightharpoonup m &= (m_1.s_0) \rightharpoonup (s_1 m_0)\\
&= (s_1 m_0)_0 \leftharpoonup [S_H((s_1 m_0)_1)(m_1.s_0)]\\
&= (s_1 m_{00}) \leftharpoonup [S_H(m_{01})(m_1.s_0)]\\
&= (s_1 m) \leftharpoonup s_0,
\end{aligned}
$$

and the condition (2.3.3) is satisfied for M.

□

Since conditions (2.3.3) and (2.3.4) are equivalent, an (S,H)-dimodule M is dyslectic if and only if the condition (2.3.4) is satisfied for M. However, we can prove this result directly as in Lemma 2.3.2. We denote by $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ the category of dyslectic (S,H)-dimodules with dyslectic (S,H)-dimodules homomorphisms; it is a full subcategory of $_S\mathscr{D}^H$.

Lemma 2.3.3. *Let M and N be dyslectic (S,H)-dimodules. Then $M\tilde{\otimes}_S N$ is a dyslectic (S,H)-dimodule.*

Proof. Suppose M and N are dyslectic (S,H)-dimodules. Let $m \in M, n \in N$ and $s \in S$. We have

$$
\begin{aligned}
(m\tilde{\otimes}n) \leftharpoonup s &= m\tilde{\otimes}(n \leftharpoonup s)\\
&= m\tilde{\otimes}(s_0 \rightharpoonup (S_H(s_1)n))\\
&= (m \leftharpoonup s_0)\tilde{\otimes}(S_H(s_1)n)\\
&= (s_{00} \rightharpoonup (S_H(s_{01})m))\tilde{\otimes}(S_H(s_1)n)\\
&= s_0 \rightharpoonup ((S_H(s_{11})m))\tilde{\otimes}(S_H(s_{12})n))\\
&= s_0 \rightharpoonup ((S_H(s_1)_2 m))\tilde{\otimes}(S_H(s_1)_1 n))\\
&= s_0 \rightharpoonup ((S_H(s_1)_1 m))\tilde{\otimes}(S_H(s_1)_2 n))\\
&= s_0 \rightharpoonup (S_H(s_1)(m\tilde{\otimes}n)).
\end{aligned}
$$

So the condition (2.3.4) is satisfied for $M\tilde{\otimes}_S N$, and $M\tilde{\otimes}_S N$ is dyslectic.

□

Theorem 2.3.4. *$(\mathscr{D}ys\text{-}_S\mathscr{D}^H, \tilde{\otimes}_S, S, \gamma)$ is a braided monoidal category.*

Proof. Let M and N be dyslectic (S,H)-dimodules. Since M and N are dyslectic, we know from Lemma 2.3.1 that $\gamma_{M,N}$ and $\gamma^{-1}_{M,N}$ are well defined. Let $m \in M, n \in N$ and $s \in S$, we have

$$
\begin{aligned}
\gamma_{M,N}(s \rightharpoonup (m\tilde{\otimes}n)) &= \gamma_{M,N}((s \rightharpoonup m)\tilde{\otimes}n)\\
&= ((s \rightharpoonup m)_1 n)\tilde{\otimes}(s \rightharpoonup m)_0\\
&= ((s_1 m_1)n)\tilde{\otimes}(s_0 \rightharpoonup m_0)\\
&= ((s_1 m_1)n \leftharpoonup s_0)\tilde{\otimes}m_0\\
&= (s \rightharpoonup (m_1 n))\tilde{\otimes}m_0\\
&= s \rightharpoonup ((m_1 n)\tilde{\otimes}m_0)\\
&= s \rightharpoonup \gamma_{M,N}(m\tilde{\otimes}n),
\end{aligned}
$$

So $\gamma_{M,N}$ left S-linear. It is clear that $\gamma_{M,N}$ is left H-linear and H-colinear and $\gamma^{-1}_{M,N}$ is its inverse.

□

Since $Dys\text{-}_{\mathscr{S}}\mathscr{D}^H$ is a braided monoidal category, by [23, Remark 4.12], every dyslectic (S,H)-dimodule finitely generated projective as a right S-module is also finitely generated projective as a left S-module.

Lemma 2.3.5. *Let M and N be dyslectic (S,H)-dimodules finitely generated projective as a right S-module. Then $Hom_S(M,N)$ and $_SHom(M,N)$ are dyslectic (S,H)-dimodules.*

Proof. Suppose M and N are dyslectic (S,H)-dimodules with M finitely generated projective as a right S-module. Let $f \in Hom_S(M,N), m \in M, n \in N$ and $s \in S$. We have

$$\begin{aligned}
(f \leftharpoonup s)(m) &= f(s \rightharpoonup m)\\
&= f((s_1.m) \leftharpoonup s_0)\\
&= f(s_1.m) \leftharpoonup s_0\\
&= s_{00} \rightharpoonup [S_H(s_{01})(f(s_1.m))]\\
&= s_0 \rightharpoonup [S_H(s_1)(f(s_2.m))]\\
&= s_0 \rightharpoonup [S_H(s_1)(f(S_H^2(s_2).m))]\\
&= s_0 \rightharpoonup [S_H(s_1)(f(S_H(S_H(s_2)).m)]\\
&= s_0 \rightharpoonup [S_H(s_{11})(f(S_H(S_H(s_{12})).m)]\\
&= [s_0 \rightharpoonup (S_H(s_1)f)](m).
\end{aligned}$$

So condition (2.3.4) is satisfied, then $Hom_S(M,N)$ is dyslectic.

Let $f \in {}_SHom(M,N), m \in M, n \in N$ and $s \in S$. We have

$$\begin{aligned}
(s \rightharpoonup f)(m) &= f(m \leftharpoonup s)\\
&= f(s_0 \rightharpoonup (S_H(s_1)m))\\
&= s_0 \rightharpoonup f(S_H(s_1)m)\\
&= [s_{01}.(f(S_H(s_1)m))] \leftharpoonup s_{00}\\
&= [s_{11}.(f(S_H(s_{12})m))] \leftharpoonup s_0\\
&= [(s_1.f)(m)] \leftharpoonup s_0\\
&= [(s_1.f) \leftharpoonup s_0](m).
\end{aligned}$$

So condition (2.3.3) is satisfied, and $_SHom(M,N)$ is dyslectic.

□

We deduce from Lemmas 2.2.4 (i), 2.3.3 and 2.3.5 that if P is finitely generated projective as a right S-module, then the functor $Hom_S(P,-)$ defined from $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ to $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ is right

adjoint to the functor $-\tilde{\otimes}_S P$ defined from $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ to $\mathscr{D}ys\text{-}_S\mathscr{D}^H$. Likewise, we deduce from Lemmas 2.2.4 (ii), 2.3.3 and 2.3.5 that if P is finitely generated projective as a left S-module, then the functor ${}_SHom(P,-)$ defined from $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ to $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ is right adjoint to the functor $-\tilde{\otimes}_S P$ defined from $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ to $\mathscr{D}ys\text{-}_S\mathscr{D}^H$.
Since $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ is a braided monoidal category, by [23, Subsection 2.2], we have an isomorphism of dyslectic (S,H)-dimodules $Hom_S(P,Q) = {}_SHom(P,Q)$ for all objects $P,Q \in \mathscr{D}ys\text{-}_S\mathscr{D}^H$ with P finitely generated projective as a left and as a right S-module. More precisely, this isomorphism is the map

$$\phi : Hom_S(P,Q) \to {}_SHom(P,Q) \quad \text{defined by} \quad \phi(f)(p) = (p_1.f)(p_0).$$

Note that in $\mathscr{D}ys\text{-}_S\mathscr{D}^H$, if N and P are finitely generated projective as right S-modules, then $N\tilde{\otimes}_S P$ is finitely generated projective as a right S-module. We know from [59] that there is a Brauer group for the braided monoidal category $\mathscr{D}ys\text{-}_S\mathscr{D}^H$. Most of the remainder of the paper is concerned with developing the details of the ingredients necessary to define this Brauer group, in a precise way.

2.4 Dyslectic (S,H)-dimodule algebras

A *dyslectic* (S,H)*-dimodule algebra* is an algebra in the braided monoidal category $\mathscr{D}ys\text{-}_S\mathscr{D}^H$, that is, an object A of $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ such that there are two dyslectic (S,H)-dimodule homomorphisms $\pi : A\tilde{\otimes}_S A \to A$ and $\mu : S \to A$ satisfying the associativity and the unitary conditions of usual algebras.
Since S is H-commutative, S is a dyslectic (S,H)-dimodule algebra. Note that a dyslectic (S,H)-dimodule algebra is an algebra in the monoidal category $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ which is dyslectic as an (S,H)-dimodule. Every H-dimodule algebra is a dyslectic (R,H)-dimodule algebra.
A *dyslectic* (S,H)*-dimodule algebra homomorphism* is a dyslectic (S,H)-dimodule homomorphism which is compatible with the product and is a unitary algebra homomorphism.

Lemma 2.4.1. *Assume that M is a dyslectic (S,H)-dimodule that is finitely generated projective as a right S-module. Then*

(i) *$End_S(M)$ is a dyslectic (S,H)-dimodule algebra: the product map is defined from $End_S(M)\tilde{\otimes}_S End_S(M)$ to $End_S(M)$ by $\pi(f\tilde{\otimes}g) = f\circ g$, for all $f,g \in End_S(M)$ and the unit map $\mu : S \to End_S(M)$ is defined by $\mu(s)(m) = s \rightharpoonup m$.*

(ii) *${}_SEnd(M)$ is a dyslectic (S,H)-dimodule algebra: the product map is defined by $\pi(f\tilde{\otimes}g) = fg = g\circ f$, for all $f,g \in {}_SEnd(M)$ and the unit map $\mu : S \to {}_SEnd(M)$ is defined by $\mu(s)(m) = s \rightharpoonup m$.*

Proof.

(i) By Lemma 2.3.5 (i), $End_S(M)$ is a dyslectic (S,H)-dimodule. For all $f,g \in End_S(M), s \in S$ and $m \in M$, we have

$$\begin{aligned}
[\pi((f \leftharpoonup s)\tilde{\otimes} g)](m) &= [(f \leftharpoonup s)\circ g](m)\\
&= (f \leftharpoonup s)(g(m))\\
&= f(s \rightharpoonup (g(m))\\
&= f((s \rightharpoonup g)(m))\\
&= (f\circ (s \rightharpoonup g))(m)\\
&= [\pi(f\tilde{\otimes}(s \rightharpoonup g))](m),
\end{aligned}$$

so π is well defined. Clearly π is left S-linear and H-linear.
For all $f,g \in End_S(M)$ and $m \in M$, we have

$$\begin{aligned}
[\pi(f\tilde{\otimes}g)]_0(m)\otimes[\pi(f\tilde{\otimes}g)]_1 &= (f\circ g)_0(m)\otimes(f\circ g)_1\\
&= [(f\circ g)(m_0)]_0\otimes[(f\circ g)(m_0)]_1 S_H(m_1)\\
&= [f(g(m_0))]_0\otimes[f(g(m_0))]_1 S_H(m_1)\\
&= [f(g(m_0)_0\varepsilon(g(m_0)_1))]_0\otimes[f(g(m_0)_0\varepsilon(g(m_0)_1))]_1 S_H(m_1)\\
&= [f(g(m_0)_0)]_0\otimes[f(g(m_0)_0)]_1\varepsilon(g(m_0)_1) S_H(m_1)\\
&= [f(g(m_0)_0)]_0\otimes[f(g(m_0)_0)]_1 S_H(g(m_0)_1)g(m_0)_2 S_H(m_1)\\
&= [f(g(m_0)_{00})]_0\otimes[f(g(m_0)_{00})]_1 S_H(g(m_0)_{01})g(m_0)_1 S_H(m_1)\\
&= f_0(g(m_0)_0)\otimes f_1 g(m_0)_1 S_H(m_1)\\
&= f_0(g_0(m))\otimes f_1 g_1\\
&= (f_0\circ g_0)(m)\otimes f_1 g_1\\
&= [\pi(f_0\tilde{\otimes}g_0)](m)\otimes f_1 g_1.
\end{aligned}$$

So π is H-colinear.
It is easy to see that μ is well defined, left S-linear and H-linear. Let us show that μ is H-colinear. For all $m \in M$ and $s \in S$, we have

$$\begin{aligned}
\mu(s)_0(m)\otimes\mu(s)_1 &= [\mu(s)(m_0)]_0\otimes[\mu(s)(m_0)]_1 S_H(m_1)\\
&= (s \rightharpoonup m_0)_0\otimes(s \rightharpoonup m_0)_1 S_H(m_1)\\
&= (s_0 \rightharpoonup m_{00})\otimes(s_1 m_{01})S_H(m_1)\\
&= (s_0 \rightharpoonup m_0)\otimes s_1\varepsilon(m_1)\\
&= (s_0 \rightharpoonup m)\otimes s_1\\
&= \mu(s_0)(m)\otimes s_1.
\end{aligned}$$

So μ is H-colinear. The identity element id_M of $End_S(M)$ is H-invariant and H-coinvariant. It is well known that the composition law is associative. Then $End_S(M)$ is an (S,H)-dimodule algebra. Therefore $End_S(M)$ is a dyslectic (S,H)-dimodule algebra.

□

Let A be a dyslectic (S,H)-dimodule algebra. The H-opposite algebra $\bar{A}$ of A is defined as follows: $\bar{A} = A$ as a dyslectic (S,H)-dimodule, but with multiplication $m_A\circ\gamma$, where m_A is the multiplication of A . In other words,

$$\bar{a}\bar{a'} = \overline{(a_1.a')a_0},\ \text{for all } a,a' \in A. \tag{2.4.1}$$

The action of S on $\bar{A}$ is given by $s \rightharpoonup \bar{a} = \overline{s \rightharpoonup a}$, the H- action by $h.\bar{a} = \overline{h.a}$, and the H-coaction by $(\bar{a})_0 \otimes (\bar{a})_1 = \bar{a}_0 \otimes a_1$, for all $a \in A$, $h \in H$ and $s \in S$. If the action of H or the coaction of H is trivial, then $\bar{A} = A^{op}$, the ordinary opposite algebra of A. Note that $\bar{S} \cong S$ when S is H-commutative.

Lemma 2.4.2. *Suppose that A is a dyslectic (S,H)-dimodule algebra. Then $\bar{A}$ is a dyslectic (S,H)-dimodule algebra.*

Proof. We need to show that the action of S is compatible with the product. Let $s \in S, a, b \in A$. Then

$$\begin{aligned}
(\bar{a} \leftharpoonup s)\bar{b} &= \overline{(a \leftharpoonup s)}\ \bar{b} \\
&= \overline{((a \leftharpoonup s)_1.b)(a \leftharpoonup s)_0} \\
&= \overline{((a_1 s_1).b)(a_0 \leftharpoonup s_0)} \\
&= \overline{[((a_1 s_1).b)a_0]} \leftharpoonup s_0 \\
&= \overline{[(a_1(s_1.b))a_0]} \leftharpoonup s_0 \\
&= \bar{a}[\overline{(s_1.b)} \leftharpoonup s_0] \\
&= \bar{a}[(s_1.\bar{b}) \leftharpoonup s_0] \\
&= \bar{a}(s \rightharpoonup \bar{b}),
\end{aligned}$$

so the multiplication in $\bar{A}$ is well defined. On the other hand, we have :

$$\begin{aligned}
(s \rightharpoonup \bar{a})\bar{b} &= \overline{(s \rightharpoonup a)}\ \bar{b} \\
&= \overline{((s \rightharpoonup a)_1.b)(s \rightharpoonup a)_0} \\
&= \overline{((s_1 a_1).b)(s_0 \rightharpoonup a_0)} \\
&= \overline{(s_1(a_1.b)) \leftharpoonup s_0)a_0} \\
&= \overline{(s \rightharpoonup (a_1.b))a_0} \\
&= \overline{s \rightharpoonup ((a_1.b)a_0)} \\
&= s \rightharpoonup \overline{((a_1.b)a_0)} \\
&= s \rightharpoonup (\bar{a}\bar{b}).
\end{aligned}$$

So the multiplication in $\bar{A}$ is S-linear. For all $a, b \in A$ and $h \in H$, we have $h.(\bar{a}\bar{b}) = (h_1.\bar{a})(h_2.\bar{b})$ and $(\bar{a}\bar{b})_0 \otimes (\bar{a}\bar{b})_1 = \bar{a}_0\bar{b}_0 \otimes \bar{a}_1\bar{b}_1$, that is, the multiplication in $\bar{A}$ is H-linear and H-colinear. Furthermore, H acts and coacts trivially on the identity element of $\bar{A}$.

□

If A and B are dyslectic (S,H)-dimodule algebras, we define a new multiplication in $A\tilde{\otimes}_S B$ by

$$(A\tilde{\otimes}_S B)\tilde{\otimes}_S(A\tilde{\otimes}_S B) \overset{1\otimes\gamma\otimes 1}{\longrightarrow} (A\tilde{\otimes}_S A)\tilde{\otimes}_S(B\tilde{\otimes}_S B) \overset{m_A\otimes m_B}{\longrightarrow} A\tilde{\otimes}_S B.$$

$A\tilde{\otimes}_S B$ with this new multiplication will be denoted $A\#_S B$, in other words,

$$(a\#b)(a'\#b') = a(b_1.a')\#b_0 b', \text{ for all } a, a' \in A;\ b, b' \in B. \tag{2.4.2}$$

$A\#_S B$ is called the ***braided product*** of A and B.

Proposition 2.4.3. *Let A, B and C be dyslectic (S,H)-dimodule algebras. Then*

(i) *$A\#_S B$ is a dyslectic (S,H)-dimodule algebra whose identity element is $1_A\#1_B$. The actions of H and S are respectively given by*

$$h.(a\#b) = (h_1.a)\#(h_2.b), \qquad \text{and} \qquad s \rightharpoonup (a\#b) = (s \rightharpoonup a)\#b \tag{2.4.3}$$

for all $h \in H, a \in A, b \in B, s \in S$ and the coaction of H is given by

$$(a\#b)_0 \otimes (a\#b)_1 = (a_0\#b_0) \otimes a_1 b_1, \qquad \forall a \in A, b \in B. \tag{2.4.4}$$

(ii) *The canonical maps*

$$\begin{array}{rrcl} \lambda_A : & A & \longrightarrow & A\#_S B \\ & a & \longmapsto & a\#1_B \end{array} \quad \text{and} \quad \begin{array}{rrcl} \lambda_B : & B & \longrightarrow & A\#_S B \\ & b & \longmapsto & 1_A\#b, \end{array}$$

are homomorphisms of dyslectic (S,H)-dimodule algebras.

(iii) *The canonical maps*

$$\begin{array}{rrcl} \lambda_A : & A & \longrightarrow & A\#_S S \\ & a & \longmapsto & a\#1_S \end{array} \quad \text{and} \quad \begin{array}{rrcl} \lambda'_A : & A & \longrightarrow & S\#_S A \\ & a & \longmapsto & 1_S\#a, \end{array}$$

are isomorphisms of dyslectic (S,H)-dimodule algebras.

(iv) *The map*

$$\phi : (A\#_S B)\#_S C \to A\#_S(B\#_S C), \ \text{given by} \quad \phi((a\#b)\#c) = a\#(b\#c)$$

for all $a \in A, b \in B$ and $c \in C$ is an isomorphism of dyslectic (S,H)-dimodule algebras.

(v) *The map*

$$\phi : \bar{B}\#_S\bar{A} \to \overline{A\#_S B} \ \text{given by} \quad \phi(\bar{b}\#\bar{a}) = \overline{(b_1.a)\#b_0},$$

for all $a \in A, b \in B$, is an isomorphism of dyslectic (S,H)-dimodule algebras.

Proof. Asume A, B and C be dyslectic (S,H)-dimodule algebras.

(i) $A\#_S B$ is isomorphic to the dyslectic (S,H)-dimodule $A\tilde{\otimes}_S B$. Let $h \in H, s \in S, a, a' \in A$ and $b, b' \in B$, we have

$$\begin{aligned}
h.[(a\#b)(a'\#b')] &= h[a(b_1.a')\#(b_0b')] \\
&= h_1.(a(b_1.a'))\#(h_2.(b_0b')) \\
&= [(h_{11}.a)(h_{12}b_1.a')]\#[(h_{21}.b_0)(h_{22}.b')] \\
&= [(h_1.a)(h_2b_1.a')]\#[(h_3.b_0)(h_4.b')] \\
&= [(h_1.a)(b_1h_2.a')]\#[(h_3.b_0)(h_4.b')] \\
&= [(h_1.a)((h_3.b)_1h_2.a')]\#[(h_3.b)_0(h_4.b')] \\
&= [(h_1.a)\#(h_3.b)][(h_2.a')\#(h_4.b')] \\
&= [(h_1.a)\#(h_2.b)][(h_3.a')\#(h_4.b')] \\
&= [(h_{11}.a)\#(h_{12}.b)][(h_{21}.a')\#(h_{22}.b')] \\
&= [h_1.(a\#b)][h_2.(a'\#b')].
\end{aligned}$$

Then the algebra map is H-linear. For all $a,a' \in A,\ b,b' \in B$, we have:

$$\begin{aligned}
&\pi[(a\#b)\tilde{\otimes}(a'\#b')]_0 \otimes \pi[(a\#b)\tilde{\otimes}(a'\#b')]_1 \\
&\quad= [(a\#b)(a'\#b')]_0 \otimes [(a\#b)(a'\#b')]_1 \\
&\quad= [a(b_1.a')\#b_0b']_0 \otimes [a(b_1.a')\#b_0b']_1 \\
&\quad= [a_0(b_1.a'_0)\#b_{00}b'_0] \otimes (a_1a'_1b_{01}b'_1) \\
&\quad= [a_0(b_{01}.a'_0)\#b_{00}b'_0] \otimes (a_1a'_1b_1b'_1) \\
&\quad= [(a_0\#b_0)(a'_0\#b'_0)] \otimes [(a_1b_1)(a'_1b'_1)] \\
&\quad= [\pi((a_0\#b_0)\tilde{\otimes}(a'_0\#b'_0))] \otimes [(a_1b_1)(a'_1b'_1)] \\
&\quad= \pi[(a\#b)_0\tilde{\otimes}(a'\#b')_0] \otimes [(a\#b)_1(a'\#b')_1],
\end{aligned}$$

therefore, the algebra map is H-colinear. We also have:

$$[s \rightharpoonup (1_A\#1_B)]_0 \otimes [s \rightharpoonup (1_A\#1_B)]_1 = [s_0 \rightharpoonup (1_A\#1_B)] \otimes (s_1 1_H),$$

this means that the unit map is H-colinear. It is easy to see that the unit map is H-linear and S-linear. Therefore $A\#_S B$ is a dyslectic (S,H)-dimodule algebra.

(ii) Let $a \in A$, we have:

$$\begin{aligned}
\lambda_A(h.a) &= (h.a)\#1_B \\
&= [h_1\varepsilon(h_2).a]\#1_B \\
&= (h_1.a)\#\varepsilon(h_2)1_B \\
&= (h_1.a)\#(h_2.1_B) \\
&= h(a\#1_B) \\
&= h\lambda_A(a),
\end{aligned}$$

for every $h \in H$. That is λ_A is left H-linear. For all $s \in S$, we have:

$$\lambda_A(s \rightharpoonup a) = (s \rightharpoonup a)\#1_B = s \rightharpoonup (a\#1_B) = s \rightharpoonup \lambda_A(a),$$

so λ_A is left S-linear. For all $a,a' \in A$,

$$\begin{aligned}\lambda_A(a)\lambda_A(a') &= (a\#1_B)(a'\#1_B)\\ &= a[(1_B)_1.a']\#(1_B)_0 1_B\\ &= a[1_H.a']\#1_B 1_B\\ &= (aa')\#1_B\\ &= \lambda_A(aa'),\end{aligned}$$

then λ_A is an algebra map. Finally we have:

$$\begin{aligned}\lambda_A(a)_0\otimes\lambda_A(a)_1 &= (a\#1_B)_0\otimes(a\#1_B)_1\\ &= (a_0\#1_B)\otimes a_1 1_H\\ &= \lambda_A(a_0)\otimes a_1,\end{aligned}$$

then the map λ_A is a right H-colinear map.

We show in the same way that the map λ_B is left S-linear, left H-linear, right H-colinear and

$$\begin{aligned}\lambda_B(bb') &= 1_A\#(bb')\\ &= 1_A(\varepsilon(b_1).1_A)\#b_0b'\\ &= (1_A(b_1.1_A))\#b_0b'\\ &= (1_A\#b)(1_A\#b')\\ &= \lambda_B(b)\lambda_B(b'), \qquad \forall b,b'\in B,\end{aligned}$$

that is, λ_B is an algebra map.

(iii) From (ii), λ_A and λ'_A are both algebras morphisms of dyslectic (S,H)-dimodule algebra. Moreover, we have

$$\lambda_A^{-1}: A\#_S S\longrightarrow A,\ \ a\#s\longmapsto\ a\leftharpoonup s,$$

in fact,

$$\lambda_A^{-1}[\lambda_A(a)] = \lambda_A^{-1}(a\#1_S) = a\leftharpoonup 1_S = a, \quad \forall a\in A$$

and

$$\lambda_A[\lambda_A^{-1}(a\#s)] = \lambda(a\leftharpoonup s) = (a\leftharpoonup s)\#1_S = a\#(s\rightharpoonup 1_S) = a\#s, \quad \forall a\in A,\ s\in S.$$

For the inverse of λ'_A, we have

$$\lambda'^{-1}_A: S\#_S A\longrightarrow A,\ \ s\#a\longmapsto\ s\rightharpoonup a.$$

Then for all $a\in A$ and $s\in S$,

$$\lambda'^{-1}_A[\lambda'_A(a)] = \lambda'^{-1}_A(1_S\#a) = 1_S\rightharpoonup a = a$$

and

$$\lambda'_A[\lambda'^{-1}_A(s\#a)] = \lambda'_A(s\rightharpoonup a) = 1_S\#(s\rightharpoonup a) = (1_S\leftharpoonup s)\#a = s\#a.$$

(iv) According to (i), $(A\#_S B)\#_S C$ and $A\#_S(B\#_S C)$ are dyslectic (S,H)-dimodule algebras. We also know that the map ϕ is an isomorphism of dyslectic (S,H)-dimodules.

(v) Let $a \in A$ and $b \in B$. For all $s \in S$, we have

$$\begin{aligned}
\phi[\bar{b}\#(s \rightharpoonup \bar{a})] &= \phi[\bar{b}\#\overline{(s \rightharpoonup a)}] \\
&= \overline{b_1.(s \rightharpoonup a)\#b_0} \\
&= \overline{[(b_1.s) \rightharpoonup (b_2.a)]\#b_0} \\
&= \overline{[(b_1.s)_1.(b_2.a) \leftharpoonup (b_1.s)_0]\#b_0} \\
&= \overline{[s_1.(b_2.a) \leftharpoonup (b_1.s_0)]\#b_0} \\
&= \overline{s_1.(b_2.a)\#[(b_1.s_0) \rightharpoonup b_0]} \\
&= \overline{s_1.(b_1.a)\#(b_0 \leftharpoonup s_0)} \\
&= \overline{(b \leftharpoonup s)_1.a\#(b \leftharpoonup s)_0} \\
&= \phi[\overline{(b \leftharpoonup s)}\#\bar{a}] \\
&= \phi[(\bar{b} \leftharpoonup s)\#\bar{a}].
\end{aligned}$$

Then ϕ is well defined. For all $s \in S$, we have;

$$\begin{aligned}
\phi[s \rightharpoonup (\bar{b}\#\bar{a})] &= \phi[(s \rightharpoonup \bar{b})\#\bar{a})] \\
&= \overline{(s \rightharpoonup b)_1.a\#(s \rightharpoonup b)_0} \\
&= \overline{(s_1.b_1).a\#(s_0 \rightharpoonup b_0)} \\
&= \overline{((s_1.b_1).a \leftharpoonup s_0)\#b_0} \\
&= \overline{(s \rightharpoonup (b_1.a))\#b_0} \\
&= \overline{s \rightharpoonup ((b_1.a)\#b_0)} \\
&= s \rightharpoonup \overline{(b_1.a)\#b_0} \\
&= s \rightharpoonup \phi(\bar{b}\#\bar{a}),
\end{aligned}$$

and

$$\begin{aligned}
\phi(\bar{b}\#\bar{a})_0 \otimes \phi(\bar{b}\#\bar{a})_1 &= (\overline{(b_1.a)\#b_0})_0 \otimes (\overline{(b_1.a)\#b_0})_1 \\
&= \overline{(b_1.a)_0\#b_{00}} \otimes \overline{(b_1.a)_1 b_{01}} \\
&= \overline{(b_1.a)_0\#b_{00}} \otimes (b_1.a)_1 b_{01} \\
&= \overline{(b_{01}.a_0)\#b_{00}} \otimes a_1 b_1 \\
&= \phi(\bar{b}_0\#\bar{a}_0) \otimes a_1 b_1 \\
&= \phi((\bar{b}\#\bar{a})_0) \otimes (\bar{b}\#\bar{a})_1.
\end{aligned}$$

We see that ϕ is S-linear and H-colinear. It is easy to see that ϕ is H-linear and compatible with the product. We also have

$$\phi(1_{\bar{B}}\#1_{\bar{A}}) = \overline{((1_{\bar{B}})_1.1_{\bar{A}})\#(1_{\bar{B}})_0} = \overline{(1_H.1_{\bar{A}})\#1_{\bar{B}}} = \overline{1_{\bar{A}}\#1_{\bar{B}}}.$$

Consider the map $\psi : \overline{A\#_S B} \to \bar{B}\#_S\bar{A}$ given by $\psi(\overline{a\#b}) = \bar{b}_0\#\overline{S_H(b_1).a}$, for all $a \in A, b \in B$.

Finally, for all $a, a' \in A$ and $b, b' \in B$, we have

$$(\phi \circ \psi)(\overline{a\#b}) = \phi(\bar{b}_0\#\overline{S_H(b_1).a}) = \overline{[b_{01}(S_H(b_1).a))]\#b_{00}} = \overline{\varepsilon(b_1)a\#b_0} = \overline{a\#b}$$

$$(\psi \circ \phi)(\bar{b'}\#\bar{a'}) = \psi(\overline{b'_1.a'\#b'_0}) = \bar{b'}_{00}\#\ \overline{S_H(b'_{01})(b'_1.a')} = \bar{b'}_0\#\ \overline{\varepsilon(b'_1)a'} = \bar{b'}\#\bar{a'}$$

Therefore ϕ is a bijection with inverse ψ.

$\square$

An (S,H)-dimodule is right faithfully projective if it is finitely generated projective as a right S-module and the canonical map

$$\psi : Hom_S(P,S) \otimes_{End_S(P)} P \longrightarrow S;\ \ f\tilde{\otimes}p \longmapsto f(p)$$

is an isomorphism.

We define in a similar way a left faithfully projective (S,H)-dimodule. An (S,H)-dimodule is said to be *faithfully projective* if it is right and left faithfully projective. Since $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ is a braided monoidal category, by [23], a dyslectic (S,H)-dimodule is right faithfully projective if and only if it is left faithfully projective. So a dyslectic (S,H)-dimodule is faithfully projective if it is right faithfully projective or left faithfully projective.
It follows from Lemma 2.4.1 that $End_S(P)$ and ${}_SEnd(P)$ are dyslectic (S,H)-dimodule algebras for any faithfully projective dyslectic (S,H)-dimodule P.
For a faithfully projective dyslectic (S,H)-dimodule M, we know that the left dual ${}_SHom(M,S)$ of M and the right dual $Hom_S(M,S)$ of M coincide in $\mathscr{D}ys\text{-}_S\mathscr{D}^H$: we will denote these duals by $M^\star$, which we regard as dyslectic (S,H)-dimodules using Lemma 2.3.5. Note that $M^\star$ is faithfully projective.

Proposition 2.4.4. *Let M be a faithfully projective dyslectic (S,H)-dimodule. Then*

(i) $End_S(M) \cong {}_SEnd(M^\star)$ *as dyslectic (S,H)-dimodule algebras;*

(ii) ${}_SEnd(M) \cong End_S(M^\star)$ *as dyslectic (S,H)-dimodule algebras;*

(iii) $\overline{End_S(M)} \cong {}_SEnd(M)$ *as dyslectic (S,H)-dimodule algebras; and*

(iv) $\overline{{}_SEnd(M)} \cong End_S(M)$ *as dyslectic* (S,H)*-dimodule algebras.*

Proof.

(i) Let $f \in End_S(M)$ and $g \in M^\star = Hom_S(M,S)$. We consider the map $\phi : End_S(M) \to {}_SEnd(M^\star)$ given by $\phi(f)(g) = g \circ f$. Since f and g are S-linear, $\phi(f)(g) \in M^\star$. For all $m \in M$ and $s \in S$, we have

$$\begin{aligned}
[\phi(f)(s \rightharpoonup g)](m) &= [(s \rightharpoonup g) \circ f](m) \\
&= (s \rightharpoonup g)(f(m)) \\
&= s \rightharpoonup [g(f(m))] \\
&= s \rightharpoonup [(g \circ f)(m)] \\
&= [s \rightharpoonup (g \circ f)](m) \\
&= [s \rightharpoonup \phi(f)(g)](m),
\end{aligned}$$

so $\phi(f)$ is left S-linear, that is $\phi(f) \in {}_SEnd(M^\star)$. Therefore $\phi(f)$ is well defined.
Consider $f \in End_S(M)$ and $g \in M^\star$. Clearly ϕ is left S-linear. For every $m \in M$, we have

$$\begin{aligned}
&[\phi(f)_0(g)](m) \otimes \phi(f)_0(g) \\
&\quad = [\phi(f)(g_0)]_0(m) \otimes [\phi(f)(g_0)]_1 S_H(g_1) \\
&\quad = (g_0 \circ f)_0(m) \otimes (g_0 \circ f)_1 S_H(g_1) \\
&\quad = [(g_0 \circ f)(m_0)]_0 \otimes [(g_0 \circ f)(m_0)]_1 S_H(g_1) S_H(m_1) \\
&\quad = [g_0(f(m_0))]_0 \otimes [g_0(f(m_0))]_1 S_H(g_1) S_H(m_1) \\
&\quad = [g(f(m_0)_0)]_{00} \otimes [g(f(m_0)_0)]_{01} S_H[[g(f(m_0)_1)]_1 S_H(f(m_0)_1)] S_H(m_1) \\
&\quad = [g(f(m_0)_0)]_{00} \otimes [g(f(m_0)_0)]_{01} S_H([g(f(m_0)_1)]_1) S_H^2(f(m_0)_1) S_H(m_1) \\
&\quad = g(f(m_0)_0) \otimes f(m_0)_1 S_H(m_1) \\
&\quad = g(f_0(m)) \otimes f_1 \\
&\quad = [\phi(f_0)(g)](m) \otimes f_1.
\end{aligned}$$

This means that ϕ is H-colinear. For all $f, f' \in End_S(M)$ and $g \in M^\star$, we have

$$\phi(ff')(g) = g \circ (ff') = (g \circ f) \circ f' = \phi(f')(g \circ f) = \phi(f')(\phi(f)(g)) = [\phi(f)\phi(f')](g).$$

So ϕ is an algebra map.
Let $\{m^{(i)}, f^{(i)}\}$ be dual bases for the S-modules M and $M^\star$, where $m^{(i)} \in M$ and $f^{(i)} \in M^\star = Hom_S(M,S)$. Then for every $m \in M$, we have $m = \sum m^{(i)} \leftharpoonup f^{(i)}(m)$.
Define the map

$$\psi : {}_SEnd(M^\star) \to End_S(M), \text{ by } \psi(g)(m) = \sum m^{(i)} \leftharpoonup [g(f^{(i)})](m).$$

Since $m = \sum m^{(i)} \leftharpoonup f^{(i)}(m)$, we have

$$f'(m) = \sum f'(m^{(i)})f^{(i)}(m) = \sum [f'(m^{(i)}) \rightharpoonup f^{(i)}](m), \text{ for every } f' \in M^\star \text{ and } m \in M.$$

So $f' = \sum f'(m^{(i)}) \rightharpoonup f^{(i)}$. For every $g \in {}_SEnd(M^\star)$, we have $g(f') = \sum f'(m^{(i)}) \rightharpoonup [g(f^{(i)})]$. This proves that $\phi \circ \psi$ is the identity map of ${}_SEnd(M^\star)$. In the similar way, we show that $\psi \circ \phi$ is the identity map of $End_S(M)$. So the algebra map ϕ is a bijection with inverse ψ.

(ii) The proof is similar to (i).

(iii) Define $\phi : \overline{End_S(M)} \to {}_SEnd(M)$ by $\phi(\bar{f})(m) = (m_1.f)(m_0)$ for all $m \in M$ and $f \in End_S(M)$. We know that ϕ is an isomorphism of dyslectic (S,H)-dimodules. Its inverse $\phi^{-1} : {}_SEnd(M) \to \overline{End_S(M)}$ is given by

$$\phi^{-1}(f)(m) = (S_H(m_1).f)(m_0), \text{ for all } m \in M, f \in {}_SEnd(M).$$

Now let show that ϕ is an algebra map. For $f, f' \in End_S(M)$ and $m \in M$, we have

$$\begin{aligned}
\phi(\bar{f}\bar{f'})(m) &= \phi(\overline{(f_1.f')f_0})(m)\\
&= [m_1.((f_1.f')f_0)](m_0)\\
&= [((m_1f_1).f')(m_2.f_0)](m_0)\\
&= ((m_1f_1).f')[m_2(f_0(S_H(m_3)m_0))]\\
&= ((m_1f((S_H(m_3)m_0)_0)_1S_H((S_H(m_3)m_0)_1)).f')[m_2(f((S_H(m_3)m_0)_0)_0)]\\
&= ((m_1f(S_H(m_3)m_{00})_1S_H(m_{01}).f')[m_2(f(S_H(m_3)m_{00})_0)]\\
&= ((S_H(m_{11})m_{12}f(S_H(m_3)m_0)_1.f')[m_2(f(S_H(m_3)m_0)_0)]\\
&= ((f(S_H(m_2)m_0)_1.f')[m_1(f(S_H(m_2)m_0)_0)]\\
&= ((m_1f(S_H(m_2)m_0)_1.f')[(m_1(f(S_H(m_2)m_0))_0]\\
&= \phi(\bar{f'})[m_1f(S_H(m_2)m_0)]\\
&= \phi(\bar{f'})[(m_1.f)(m_0)]\\
&= \phi(\bar{f'})[\phi(\bar{f})(m)]\\
&= [\phi(\bar{f})\phi(\bar{f'})](m).
\end{aligned}$$

Hence, ϕ is an isomorphism of dyslectic (S,H)-dimodule algebras.

(iv) It can be proved like (iii).

□

If M and N are faithfully projective dyslectic (S,H)-dimodules, then $M\tilde{\otimes}_SN$ is a faithfully projective dyslectic (S,H)-dimodule.

Proposition 2.4.5. *Let M and N be faithfully projective dyslectic (S,H)-dimodules. Then*

$$End_S(M)\#_S\, End_S(N) \cong End_S(M\tilde{\otimes}_S N) \quad and \quad {}_SEnd(M)\#_S\, {}_SEnd(N) \cong {}_SEnd(M\tilde{\otimes}_S N)$$

as dyslectic (S,H)-dimodule algebras.

Proof. Define the map $\phi : End_S(M)\#_S\, End_S(N) \to End_S(M\tilde{\otimes}_S N)$ by

$$\phi(f\#g)(m\tilde{\otimes}n) = f(g_1 m)\tilde{\otimes}g_0(n), \text{ for all } f \in End_S(M),\ g \in End_S(N), m \in M \text{ and } n \in N.$$

It is clear that $\phi(f\#g)$ and ϕ are well defined left H-linear maps. ϕ is an H-colinear map since for all $f \in End_S(M), g \in End_S(N), m \in M$ and $n \in N$, we have

$$\begin{aligned}
&\phi(f\#g)_0(m\tilde{\otimes}n)\otimes\phi(f\#g)_1\\
&\quad = [\phi(f\#g)((m\tilde{\otimes}n)_0)]_0 \otimes [\phi(f\#g)((m\tilde{\otimes}n)_0)]_1 S_H((m\tilde{\otimes}n)_1)\\
&\quad = [\phi(f\#g)(m_0\tilde{\otimes}n_0)]_0 \otimes [\phi(f\#g)(m_0\tilde{\otimes}n_0)]_1 S_H(m_1n_1)\\
&\quad = [f(g_1m_0)\tilde{\otimes}g_0(n_0)]_0 \otimes [f(g_1m_0)\tilde{\otimes}g_0(n_0)]_1 S_H(m_1n_1)\\
&\quad = [f[(g(n_{00})_1S_H(n_{01}))m_0]\tilde{\otimes}g(n_{00})_0]_0 \otimes [f[(g(n_{00})_1S_H(n_{01}))m_0]\tilde{\otimes}g(n_{00})_0]_1 S_H(m_1n_1)\\
&\quad = [f[(g(n_{00})_1S_H(n_{01}))m_0]_0\tilde{\otimes}g(n_{00})_{00}] \otimes [f[(g(n_{00})_1S_H(n_{01}))m_0]_1 g(n_{00})_{01}] S_H(m_1)S_H(n_1)\\
&\quad = [f[(g(n_{00})_{01}S_H(n_{01}))m_0]_0\tilde{\otimes}g(n_{00})_{00}] \otimes [f[(g(n_{00})_{01}S_H(n_{01}))m_0]_1 g(n_{00})_1] S_H(m_1)S_H(n_1)\\
&\quad = [f[((g(n_{00})_{01}S_H(n_{01}))m)_0]_0\tilde{\otimes}g(n_{00})_{00}] \otimes [f[((g(n_{00})_{01}S_H(n_{01}))m)_0]_1 g(n_{00})_1]\\
&\qquad \times S_H((g(n_{00})_{01}S_H(n_{01}))m)_1)S_H(n_1)\\
&\quad = [f_0[(g(n_{00})_{01}S_H(n_{01}))m]\tilde{\otimes}g(n_{00})_{00}] \otimes f_1 g(n_{00})_1 S_H(n_1)\\
&\quad = [f_0[(g(n_0)_{01}S_H(n_{11}))m]\tilde{\otimes}g(n_0)_{00}] \otimes f_1 g(n_0)_1 S_H(n_{12})\\
&\quad = [f_0[(g_0(n_0)_1S_H(n_{11}))m]\tilde{\otimes}g_0(n_0)_0] \otimes f_1g_1\\
&\quad = [f_0(g_{01}m)\tilde{\otimes}g_{00}(n)] \otimes f_1g_1\\
&\quad = \phi(f_0\#g_0)(m\tilde{\otimes}n) \otimes f_1g_1\\
&\quad = \phi((f\#g)_0)(m\tilde{\otimes}n) \otimes (f\#g)_1.
\end{aligned}$$

Now let us show that ϕ is an algebra map. We have

$$\begin{aligned}
\phi[(f\#g)(f'\#g')](m\tilde{\otimes}n) &= \phi[f(g_1.f')\#g_0g'](m\tilde{\otimes}n)\\
&= (f(g_1.f'))[(g_0g')_1m]\tilde{\otimes}(g_0g')_0(n)\\
&= f[(g_1.f')(g_{01}g'_1m)]\tilde{\otimes}(g_{00}g'_0)(n)\\
&= f[(g_2.f')(g_1g'_1m)]\tilde{\otimes}(g_0g'_0)(n)\\
&= f[(g_1.f')(g_2g'_1m)]\tilde{\otimes}(g_0g'_0)(n)\\
&= f[g_{11}f'(S_H(g_{12})g_2g'_1m)]\tilde{\otimes}g_0(g'_0(n))\\
&= f[g_1f'(g'_1m)]\tilde{\otimes}g_0(g'_0(n))\\
&= \phi(f\#g)[f'(g'_1m)\tilde{\otimes}g'_0(n)]\\
&= \phi(f\#g)[\phi(f'\#g')(m\tilde{\otimes}n)]\\
&= [\phi(f\#g)\phi(f'\#g')](m\tilde{\otimes}n)
\end{aligned}$$

for all $f \in End_S(M), g \in End_S(N), m \in M$ and $n \in N$. Then ϕ is a homomorphism of (S,H)-dimodules algebras. Our assumptions imply that every element of $End_S(M\tilde{\otimes}_S N)$ has the form $f\tilde{\otimes}g$, for $f \in End_S(M)$ and $g \in End_S(N)$. Let define the map

$$\psi : End_S(M\tilde{\otimes}_S N) \to End_S(M)\#_S End_S(N), \text{ by } \psi(f\tilde{\otimes}g)(m\tilde{\otimes}n) = f(S_H(g_1)m)\tilde{\otimes}g_0(n).$$

ϕ is a bijection with inverse ψ. Therefore, ϕ is an isomorphism of dyslectic (S,H)-dimodule algebras.

The proof of the second isomorphism follows from Proposition 2.4.4 (ii).

□

Lemma 2.4.6. *Let M be a faithfully projective dyslectic (S,H)-dimodule. Then*

(i) *$M\tilde{\otimes}_S M^\star$ is a faithfully projective dyslectic (S,H)-dimodule algebra: the multiplication in $M\tilde{\otimes}_S M^\star$ is defined by*

$$(m\tilde{\otimes}f)(m'\tilde{\otimes}f') = [m \leftharpoonup f(m')]\tilde{\otimes}f', \text{ for all } m,m' \in M,\ f,f' \in M^\star. \tag{2.4.5}$$

(ii) *The natural S-linear map $\phi : M\tilde{\otimes}_S M^\star \cong End_S(M)$ defined by*

$$\phi(m\tilde{\otimes}f)(m') = m \leftharpoonup f(m'), \text{ for all } m,m' \in M,\ f \in M^\star,$$

is an isomorphism of dyslectic (S,H)-dimodule algebras.

Proof.

(i) We know that $M\tilde{\otimes}_S M^\star$ is a faithfully projective dyslectic (S,H)-dimodule since M and $M^\star$ are faithfully projective dyslectic (S,H)-dimodules. Let $s \in S$, $m,n \in M$ and $f,g \in M^\star$, we have

$$\begin{aligned}
[(m\tilde{\otimes}f) \leftharpoonup s](n\tilde{\otimes}g) &= [m\tilde{\otimes}(f \leftharpoonup s)](n\tilde{\otimes}g)\\
&= [m \leftharpoonup (f \leftharpoonup s)(n)]\tilde{\otimes}g\\
&= [m \leftharpoonup f(s \rightharpoonup n)]\tilde{\otimes}g\\
&= (m\tilde{\otimes}f)[(s \rightharpoonup n)\tilde{\otimes}g]\\
&= (m\tilde{\otimes}f)[s \rightharpoonup (n\tilde{\otimes}g)],
\end{aligned}$$

then the multiplication of $M\tilde{\otimes}_S M^\star$ is well defined. For all $h \in H$, $m,n \in M$ and $f,g \in M^\star$, we have

$$\begin{aligned}
[s \rightharpoonup (m\tilde{\otimes}f)](n\tilde{\otimes}g) &= [(s \rightharpoonup m)\tilde{\otimes}f](n\tilde{\otimes}g)\\
&= [(s \rightharpoonup m) \leftharpoonup f(n)]\tilde{\otimes}g\\
&= [s \rightharpoonup (m \leftharpoonup f(n))]\tilde{\otimes}g\\
&= s \rightharpoonup [(m \leftharpoonup f(n))\tilde{\otimes}g]\\
&= s \rightharpoonup [(m\tilde{\otimes}f)(n\tilde{\otimes}g).,
\end{aligned}$$

We also have $h[(m\tilde{\otimes}f)(n\tilde{\otimes}g)] = [h_1(m\tilde{\otimes}f)][h_2(n\tilde{\otimes}g)]$ and

$$\begin{aligned}
[(m\tilde{\otimes}f)(n\tilde{\otimes}g)]_0 \otimes [(m\tilde{\otimes}f)(n\tilde{\otimes}g)]_1 &= [(m \leftharpoonup f(n))\tilde{\otimes}g]_0 \otimes [(m \leftharpoonup f(n))\tilde{\otimes}g]_1\\
&= [(m \leftharpoonup f(n))_0\tilde{\otimes}g_0] \otimes [(m \leftharpoonup f(n))_1 g_1]\\
&= [(m_0 \leftharpoonup f(n)_0)\tilde{\otimes}g_0] \otimes [m_1 f(n)_1 g_1]\\
&= [(m_0 \leftharpoonup f(n_0)_0)\tilde{\otimes}g_0] \otimes [m_1 f(n_0)_1 \varepsilon(n_1) g_1]\\
&= [(m_0 \leftharpoonup f(n_{00})_0)\tilde{\otimes}g_0] \otimes [m_1 f(n_{00})_1 S_H(n_0 1) n_1 g_1]\\
&= [(m_0 \leftharpoonup f_0(n_0))\tilde{\otimes}g_0] \otimes [m_1 f_1 n_1 g_1]\\
&= [(m_0\tilde{\otimes}f_0)(n\tilde{\otimes}g_0] \otimes [(m_1 f_1)(n_1 g_1)]\\
&= (m\tilde{\otimes}f)_0(n\tilde{\otimes}g)_0 \otimes (m\tilde{\otimes}f)_1(n\tilde{\otimes}g)_1.
\end{aligned}$$

So the left S-action, the left H-action and the coaction are compatible with the product of $M\tilde{\otimes}_S M^\star$.
This product is associative since,

$$\begin{aligned}
[(m\tilde{\otimes}f)(n\tilde{\otimes}g)](p\tilde{\otimes}l) &= [(m \leftharpoonup f(n))\tilde{\otimes}g](p\tilde{\otimes}l)\\
&= [(m \leftharpoonup f(n)) \leftharpoonup g(p)]\tilde{\otimes}l\\
&= [m \leftharpoonup (f(n) \leftharpoonup g(p))]\tilde{\otimes}l\\
&= [m \leftharpoonup (f(n \leftharpoonup g(p)))]\tilde{\otimes}l\\
&= (m\tilde{\otimes}f)[(n \leftharpoonup g(p))\tilde{\otimes}l]\\
&= (m\tilde{\otimes}f)[(n\tilde{\otimes}g)(p\tilde{\otimes}l)],
\end{aligned}$$

for all $m,n,p \in M$ and $f,g,l \in M^\star$. So $M\tilde{\otimes}_S M^\star$ is an (S,H)-dimodule algebra.

(ii) Since M is faithfully projective, ϕ is an isomorphism of dyslectic (S,H)-dimodules. Now let us show that ϕ preserves the product and the identity element of $M\tilde{\otimes}_S M^\star$. For all $m,m',m'' \in M$ and $f,f' \in M^\star$,

$$\begin{aligned}
\phi[(m\tilde{\otimes}f)(m'\tilde{\otimes}f')](m'') &= \phi[(m \leftharpoonup f(m'))\tilde{\otimes}f'](m'')\\
&= (m \leftharpoonup f(m')) \leftharpoonup f'(m'')\\
&= m \leftharpoonup [f(m')f'(m'')]\\
&= m \leftharpoonup [f(m' \leftharpoonup f(m''))],\\
&= m \leftharpoonup f(m' \leftharpoonup f'(m''))\\
&= m \leftharpoonup f[\phi(m'\tilde{\otimes}f')(m'')]\\
&= \phi(m\tilde{\otimes}f)[\phi(m'\tilde{\otimes}f')(m'')]\\
&= [\phi(m\tilde{\otimes}f)\phi(m'\tilde{\otimes}f')](m'').
\end{aligned}$$

So ϕ preserves the product. Let $\{m^{(i)}\}$ and $\{f^{(i)}\}$ be the dual bases of M and $M^\star$. We have

$$\phi(\sum m^{(i)}\tilde{\otimes}f^{(i)})(m') = \sum m^{(i)} \leftharpoonup f^{(i)}(m') = m'.$$

We deduce that $\phi(\sum m^{(i)}\tilde{\otimes}f^{(i)}) = id_{End_S(M)}$. So ϕ preserves the identity element $m^{(i)}\tilde{\otimes}f^{(i)}$ of $M\tilde{\otimes}_S M^\star$.

□

Lemma 2.4.7. *Let M be a faithfully projective dyslectic (S,H)-dimodule. Then*

(i) *$M^\star\tilde{\otimes}_S M$ is a faithfully projective dyslectic (S,H)-dimodule algebra: the multiplication in $M^\star\tilde{\otimes}_S M$ is defined by*

$$(f\tilde{\otimes}m)(f'\tilde{\otimes}m') = f\tilde{\otimes}[f'(m) \rightharpoonup m'] \text{ for all } m,m' \in M,\ f,f' \in M^\star. \tag{2.4.6}$$

(ii) *The natural R-linear map $\phi : M^\star\tilde{\otimes}_S M \cong {}_SEnd(M)$ defined by*

$$\phi(f\tilde{\otimes}m)(m') = f(m') \rightharpoonup m, \text{ for all } m,m' \in M,\ f \in {}_SEnd(M),$$

is an isomorphism of dyslectic (S,H)-dimodule algebras.

Proof.

(i) We know that $M^\star\tilde{\otimes}_S M$ is a faithfully projective dyslectic (S,H)-dimodule since M and $M^\star = {}_SHom(M,S)$ are faithfully projective dyslectic (S,H)-dimodules. Let $s \in S$, $m,m' \in M$ and $f,f' \in M^\star$, we have

$$\begin{aligned}
[(f\tilde{\otimes}m) \leftharpoonup s](f'\tilde{\otimes}m') &= [f\tilde{\otimes}(m \leftharpoonup s)](f'\tilde{\otimes}m')\\
&= f\tilde{\otimes}[f'(m \leftharpoonup s) \rightharpoonup m']\\
&= f\tilde{\otimes}[(s \rightharpoonup f')(m) \rightharpoonup m']\\
&= (f\tilde{\otimes}m)[(s \rightharpoonup f')\tilde{\otimes}m']\\
&= (f\tilde{\otimes}m)[s \rightharpoonup (f'\tilde{\otimes}m')].
\end{aligned}$$

So the product of $M^\star \tilde{\otimes}_S M$ is well defined. For all $s \in S$, $m, m' \in M$ and $f, f' \in M^\star$, we have

$$\begin{aligned}
[s \rightharpoonup (f\tilde{\otimes}m)](f'\tilde{\otimes}m') &= [(s \rightharpoonup f)\tilde{\otimes}m](f'\tilde{\otimes}m') \\
&= (s \rightharpoonup f)\tilde{\otimes}(f'(m) \rightharpoonup m') \\
&= s \rightharpoonup [f\tilde{\otimes}(f'(m) \rightharpoonup m')] \\
&= s \rightharpoonup [(f\tilde{\otimes}m)(f'\tilde{\otimes}m')].
\end{aligned}$$

Let $h \in H$,

$$\begin{aligned}
h[(f\tilde{\otimes}m)(f'\tilde{\otimes}m')] &= h[f\tilde{\otimes}(f'(m) \rightharpoonup m')] \\
&= (h_1.f)\tilde{\otimes}[h_2(f'(m)) \rightharpoonup (h_3m')] \\
&= (h_1.f)\tilde{\otimes}[h_2(f'(\varepsilon(h_3m)) \rightharpoonup (h_4m')] \\
&= (h_1.f)\tilde{\otimes}[h_2(f'(S_H(h_{31})h_{32}m)) \rightharpoonup (h_4m')] \\
&= (h_1.f)\tilde{\otimes}[h_{21}(f'(S_H(h_{22})h_3m)) \rightharpoonup (h_4m')] \\
&= (h_1.f)\tilde{\otimes}[(h_2.f')(h_3m) \rightharpoonup (h_4m')] \\
&= (h_1.f)\tilde{\otimes}[(h_3.f')(h_2m) \rightharpoonup (h_4m')] \\
&= [(h_1.f)\tilde{\otimes}(h_2m)][(h_3.f')\tilde{\otimes}(h_4m')] \\
&= [h_1(f\tilde{\otimes}m)][h_2(f'\tilde{\otimes}m')].
\end{aligned}$$

We also have

$$\begin{aligned}
&[(f\tilde{\otimes}m)(f'\tilde{\otimes}m')]_0 \otimes [(f\tilde{\otimes}m)(f'\tilde{\otimes}m')]_1 \\
&\quad = [f\tilde{\otimes}(f'(m) \rightharpoonup m')]_0 \otimes [f\tilde{\otimes}(f'(m) \rightharpoonup m')]_1 \\
&\quad = [f_0\tilde{\otimes}(f'(m) \rightharpoonup m')_0] \otimes [f_1(f'(m) \rightharpoonup m')_1] \\
&\quad = [f_0\tilde{\otimes}(f'(m)_0 \rightharpoonup m'_0)] \otimes f_1 f'(m)_1 m'_1 \\
&\quad = [f_0\tilde{\otimes}(f'(m_0)_0 \rightharpoonup m'_0)] \otimes f_1 f'(m_0)_1 \varepsilon(m_1) m'_1 \\
&\quad = [f_0\tilde{\otimes}(f'(m_0)_0 \rightharpoonup m'_0)] \otimes f_1 f'(m_0)_1 S_H(m_1) m_2 m'_1 \\
&\quad = [f_0\tilde{\otimes}(f'(m_{00})_0 \rightharpoonup m'_0)] \otimes f_1 f'(m_{00})_1 S_H(m_{01}) m_1 m'_1 \\
&\quad = [f_0\tilde{\otimes}(f'_0(m_0) \rightharpoonup m'_0)] \otimes f_1 f'_1 m_1 m'_1 \\
&\quad = (f_0\tilde{\otimes}m_0)(f'_0\tilde{\otimes}m'_0) \otimes f_1 m_1 f'_1 m'_1 \\
&\quad = (f\tilde{\otimes}m)_0(f'\tilde{\otimes}m')_0 \otimes (f\tilde{\otimes}m)_1(f'\tilde{\otimes}m')_1.
\end{aligned}$$

So the left S-action, the left H-action and the coaction are compatible with the product of $M^\star \tilde{\otimes}_S M$.

This product is also associative since,

$$\begin{aligned}
[(f\tilde{\otimes}m)(f'\tilde{\otimes}m'](f''\tilde{\otimes}m'') &= [f\tilde{\otimes}(f'(m)\rightharpoonup m')](f''\tilde{\otimes}m'')\\
&= f\tilde{\otimes}[f''(f'(m)\rightharpoonup m')\rightharpoonup m'']\\
&= f\tilde{\otimes}[f'(m)f''(m')\rightharpoonup m'']\\
&= f\tilde{\otimes}[f'(m)\rightharpoonup (f''(m')\rightharpoonup m'')]\\
&= (f\tilde{\otimes}m)[f'\tilde{\otimes}(f''(m')\rightharpoonup m'')]\\
&= (f\tilde{\otimes}m)[(f'\tilde{\otimes}m')(f''\tilde{\otimes}m'')]
\end{aligned}$$

for all $f,f',f''\in M^\star$, and $m,m',m''\in M$. Therefore $M^\star\tilde{\otimes}_S M$ is an (S,H)-dimodule algebra.

(ii) As lemma 2.4.6 we will just show that ϕ preserves the product and the identity element of $M^\star\tilde{\otimes}_S M$. For every $f,f'\in M^\star$ and $m,m',m''\in M$, we have

$$\begin{aligned}
\phi[(f\tilde{\otimes}m)(f'\tilde{\otimes}m')](m'') &= \phi[f\tilde{\otimes}(f'(m)\rightharpoonup m)](m'')\\
&= f(m'')\rightharpoonup [f'(m)\rightharpoonup m']\\
&= [f(m'')f'(m)]\rightharpoonup m'\\
&= [f'(f(m'')\rightharpoonup m)]\rightharpoonup m'\\
&= \phi(f'\tilde{\otimes}m')(f(m'')\rightharpoonup m)\\
&= \phi(f'\tilde{\otimes}m')[\phi(f\tilde{\otimes}m)(m'')]\\
&= [\phi(f\tilde{\otimes}m)\phi(f'\tilde{\otimes}m')](m'')
\end{aligned}$$

Note $f^{(i)}\tilde{\otimes}m^{(i)}$ is the identity element of $M^\star\tilde{\otimes}_S M$, where $\{f^{(i)}\}$ and $\{m^{(i)}\}$ are respectively the dual bases of $M^\star$ and M. Hence for every $m'\in M$ we have

$$\phi(\sum f^{(i)}\tilde{\otimes}m^{(i)})(m') = \sum f^{(i)}(m')\rightharpoonup m^{(i)} = m' = id_{{}_S End(M)}(m').$$

Proposition 2.4.8. *Let A be a dyslectic (S,H)-dimodule algebra. If M is a dyslectic (S,H)-dimodule that is faithfully projective as an S-module, then*

$$A\#_S End_S(M)\cong End_S(M)\#_S A \quad and \quad {}_S End(M)\#_S A\cong A\#_S {}_S End(M)$$

as dyslectic (S,H)-dimodule algebras.

Proof. This is shown for general braided monoidal categories in [59, Proposition 2.4(i)]. Here we identify $End_S(M)$ with $M\tilde{\otimes}_S M^\star$, we define the map η as the composition of the following morphisms:

$$\eta: A\#_S M\tilde{\otimes}_S M^\star \overset{\gamma^{-1}\tilde{\otimes}_S id}{\longrightarrow} M\tilde{\otimes}_S A\#_S M^\star \overset{id\tilde{\otimes}_S\gamma}{\longrightarrow} M\tilde{\otimes}_S M^\star\#_S A$$

given by

$$\eta(a\#m\tilde{\otimes}f) = m_0\tilde{\otimes}a_1.f\#S_H(m_1).a_0,\ \text{for all } a\in A, m\in M \text{ and } f\in M^\star.$$

So η is well defined. Now let us show that η is an algebra map. For all $a,b\in A$, $m,n\in M$ and $f,g\in M^\star$, we have

$$
\begin{aligned}
&\eta[(a\#m\tilde{\otimes}f)(b\#n\tilde{\otimes}g)]\\
&\quad= \eta[a((m\tilde{\otimes}f)_1.b)\#(m\tilde{\otimes}f)_0(n\tilde{\otimes}g)]\\
&\quad= \eta[a((m_1f_1).b)\#(m_0\tilde{\otimes}f_0)(n\tilde{\otimes}g)]\\
&\quad= \eta[a((m_1f_1).b)\#(m_0 \leftharpoonup f_0(n))\tilde{\otimes}g]\\
&\quad= (m_0 \leftharpoonup f_0(n))_0\tilde{\otimes}[a((m_1f_1).b)]_1.g\#S_H[(m_0 \leftharpoonup f_0(n))_1].[a((m_1f_1).b)]_0\\
&\quad= (m_0 \leftharpoonup f_0(n)_0)\tilde{\otimes}(a_1b_1).g\#S_H(m_1f_0(n)_1).[a_0((m_2f_1).b_0)]\\
&\quad= m_0\tilde{\otimes}[f_0(n)_0 \rightharpoonup (a_1b_1).g]\#S_H(m_1f_0(n)_1).[a_0((m_2f_1).b_0)]\\
&\quad= m_0\tilde{\otimes}[(f_0(n)_1a_1b_1).g \leftharpoonup f_0(n)_0]\#S_H(m_1f_0(n)_2).[a_0((m_2f_1).b_0)]\\
&\quad= m_0\tilde{\otimes}(f_0(n)_2a_1b_1).g\#[f_0(n)_0 \rightharpoonup S_H(m_1f_0(n)_1).[a_0((m_2f_1).b_0)]]\\
&\quad= m_0\tilde{\otimes}(f_0(n)_2a_1b_1).g\#[f_0(n)_0 \rightharpoonup S_H(f_0(n)_1)S_H(m_1).[a_0((m_2f_1).b_0)]]\\
&\quad= m_0\tilde{\otimes}(f_0(n)_1a_1b_1).g\#[S_H(m_1).[a_0((m_2f_1).b_0)] \leftharpoonup f_0(n)_0]\\
&\quad= m_0\tilde{\otimes}(f_0(n)_1a_1b_1).g\#[(S_H(m_1).a_0)[(S_H(m_2)(m_3f_1).b_0) \leftharpoonup f_0(n)_0]]\\
&\quad= m_0\tilde{\otimes}(f_0(n)_1a_1b_1).g\#[(S_H(m_1).a_0)[(f_1.b_0) \leftharpoonup f_0(n)_0]]\\
&\quad= m_0\tilde{\otimes}(f(n_0)_{01}a_1b_1).g\#[(S_H(m_1).a_0)[(f(n_0)_1S_H(n_1).b_0) \leftharpoonup f(n_0)_{00}]]\\
&\quad= m_0\tilde{\otimes}(f(n_0)_1a_1b_1).g\#[(S_H(m_1).a_0)[f(n_0)_0 \rightharpoonup (S_H(n_1).b_0)]]
\end{aligned}
$$

$$
\begin{aligned}
&\eta(a\#m\tilde{\otimes}f)\eta(b\#n\tilde{\otimes}g)\\
&\quad= [m_0\tilde{\otimes}a_1.f\#S_H(m_1).a_0][n_0\tilde{\otimes}b_1.g\#S_H(n_1).b_0]\\
&\quad= (m_0\tilde{\otimes}a_1.f)[(S_H(m_1).a_0)_1.(n_0\tilde{\otimes}b_1.g)]\#(S_H(m_1).a_0)_0(S_H(n_1).b_0)\\
&\quad= (m_0\tilde{\otimes}a_2.f)[a_1.(n_0\tilde{\otimes}b_1.g)]\#(S_H(m_1).a_0)(S_H(n_1).b_0)\\
&\quad= [m_0 \leftharpoonup (a_3.f)(a_1n_0)]\tilde{\otimes}a_2b_1.g\#(S_H(m_1).a_0)(S_H(n_1).b_0)\\
&\quad= [m_0 \leftharpoonup a_3.f(S_H(a_2)a_1n_0)]\tilde{\otimes}a_4b_1.g\#(S_H(m_1).a_0)(S_H(n_1).b_0)\\
&\quad= [m_0 \leftharpoonup a_1.f(n_0)]\tilde{\otimes}a_2b_1.g\#(S_H(m_1).a_0)(S_H(n_1).b_0)\\
&\quad= m_0\tilde{\otimes}[a_1.f(n_0) \rightharpoonup a_2b_1.g]\#(S_H(m_1).a_0)(S_H(n_1).b_0)\\
&\quad= m_0\tilde{\otimes}[(f(n_0)_1a_2b_1).g \leftharpoonup a_1.f(n_0)_0]\#(S_H(m_1).a_0)(S_H(n_1).b_0)\\
&\quad= m_0\tilde{\otimes}(f(n_0)_1a_2b_1).g\#[a_1.f(n_0)_0 \rightharpoonup (S_H(m_1).a_0)(S_H(n_1).b_0)]\\
&\quad= m_0\tilde{\otimes}(f(n_0)_1a_1b_1).g\#[a_{01}.f(n_0)_0 \rightharpoonup (S_H(m_1).a_{00})(S_H(n_1).b_0)]\\
&\quad= m_0\tilde{\otimes}(f(n_0)_1a_1b_1).g\#[[a_{01}.f(n_0)_0 \rightharpoonup (S_H(m_1).a_{00})](S_H(n_1).b_0)]\\
&\quad= m_0\tilde{\otimes}(f(n_0)_1a_2b_1).g\#[[(S_H(m_1).a_0)_0 \leftharpoonup S_H[(S_H(m_1).a_0)_1](a_1.f(n_0)_0)](S_H(n_1).b_0)]\\
&\quad= m_0\tilde{\otimes}(f(n_0)_1a_3b_1).g\#[[(S_H(m_1).a_0) \leftharpoonup S_H(a_1)(a_2.f(n_0)_0)](S_H(n_1).b_0)]\\
&\quad= m_0\tilde{\otimes}(f(n_0)_1a_1b_1).g\#[[(S_H(m_1).a_0) \leftharpoonup f(n_0)_0](S_H(n_1).b_0)]\\
&\quad= m_0\tilde{\otimes}(f(n_0)_1a_1b_1).g\#[(S_H(m_1).a_0)[f(n_0)_0 \rightharpoonup (S_H(n_1).b_0)]].
\end{aligned}
$$

We have $\eta[(a\#m\tilde{\otimes}f)(b\#n\tilde{\otimes}g)] = \eta(a\#m\tilde{\otimes}f)\eta(b\#n\tilde{\otimes}g)$ so η is an algebra map. We know that η is an isomorphism of (S,H)-dimodules. Its inverse is the map

$$\xi : M\tilde{\otimes}_S M^\star\#_S A \xrightarrow{id\tilde{\otimes}_S\gamma^{-1}} M\tilde{\otimes}_S A\#_S M^\star \xrightarrow{\gamma\tilde{\otimes}_S id} A\#_S M\tilde{\otimes}_S M^\star,$$

given by

$$\xi(m\tilde{\otimes}f\#a) = m_1.a_0\#m_0\tilde{\otimes}S_H(a_1).f, \text{ for all } m \in M, f \in M^\star, a \in A.$$

Therefore $A\#_S\, End_S(M) \cong End_S(M)\#_S A$ as dyslectic (S,H)-dimodule algebras.

2.5 Dyslectic (S,H)-dimodule Azumaya algebras

We will introduce the notion of a dyslectic (S,H)-dimodule Azumaya algebra and work from there toward our eventual goal of defining the Brauer-Clifford-Long group.

Proposition 2.5.1. *Let A be a dyslectic (S,H)-dimodule algebra which is faithfully projective as an S-module. We define two S-linear maps*

$$\begin{array}{lll} F: \ A\#_S\bar{A} \to End_S(A) & \quad \text{and} \quad & G: \ \bar{A}\#_S A \to \overline{End_S(A)} \\ \quad (a\#\bar{b})(c) \mapsto a(b_1.c)b_0 & & \quad (\bar{a}\#b)(c) \mapsto (c_1.a)c_0 b, \end{array}$$

for all a,b and $c \in A$. Then F and G are dyslectic (S,H)-dimodule algebra homomorphisms.

Proof. The map $F(a\#b)$ is S-linear and F is well defined since

$$\begin{aligned} F(a\#b)(c \leftharpoonup s) &= a(b_1.(c \leftharpoonup s))b_0 \\ &= a((b_1.c) \leftharpoonup (b_2.s))b_0 \\ &= a(b_2.c)((b_1.s) \rightharpoonup b_0) \\ &= a(b_1.c)(b_0 \leftharpoonup s) \\ &= [a(b_1.c)b_0] \leftharpoonup s \\ &= [F(a\#b)(c)] \leftharpoonup s, \end{aligned}$$

and

$$\begin{aligned} F[(a \leftharpoonup s)\#\bar{b}](c) &= (a \leftharpoonup s)(b_1.c)b_0 \\ &= a(s \rightharpoonup (b_1.c))b_0 \\ &= a(s_1.(b_1.c) \leftharpoonup s_0))b_0 \\ &= a((s_1b_1).c)(s_0 \rightharpoonup b_0) \\ &= a((s \rightharpoonup b)_1.c)(s \rightharpoonup b)_0 \\ &= F[a\#\overline{s \rightharpoonup b}](c) \\ &= F[a\#(s \rightharpoonup \bar{b})](c), \end{aligned}$$

for all $a,b,c \in A$ and $s \in S$. It is clear that F is H-linear and left S-linear. For the right H-colinearity, we have:

$$\begin{aligned} [F(a\#\bar{b})]_0(c) \otimes [F(a\#\bar{b})]_1 &= [F(a\#\bar{b})(c_0)] \otimes [F(a\#\bar{b})(c_0)]_1 S_H(c_1) \\ &= [a(b_1.c_0)b_0]_0 \otimes [a(b_1.c_0)b_0]_1 S_H(c_1) \\ &= a_0(b_1.c_0)_0 b_{00} \otimes a_1(b_1.c_0)_1 b_{01} S_H(c_1) \\ &= a_0(b_1.c_{00})b_{00} \otimes a_1 c_{01} b_{01} S_H(c_1) \\ &= a_0(b_{01}.c_0)b_{00} \otimes a_1 b_1 \varepsilon(c_1) \\ &= a_0(b_{01}.c_0\varepsilon(c_1))b_{00} \otimes a_1 b_1 \\ &= F(a_0\#\bar{b}_0)(c) \otimes a_1\bar{b}_1 \\ &= F((a\#\bar{b})_0)(c) \otimes (a\#\bar{b})_1. \end{aligned}$$

Finally, let us show that F is an algebra map. For all $a, a', b, b'c \in A$, we have

$$\begin{aligned}
F[(a\#\bar{b})(a'\#\bar{b}')](c) &= F[a(\bar{b}_1.a')\#\bar{b}_0\bar{b}'](c) \\
&= F[a(b_2.a')\#\overline{(b_1.b')b_0}](c) \\
&= (a(b_2.a'))[((b_1.b')b_0)_1.c]((b_1.b')b_0)_0 \\
&= (a(b_2.a'))[((b_1.b')_1 b_{01}).c]((b_1.b')_0 b_{00}) \\
&= (a(b_2.a'))[(b'_1 b_{01}).c]((b_1.b'_0)b_{00}) \\
&= a[(b_2.a')((b'_1 b_{01}).c)(b_1.b'_0)]b_{00} \\
&= a[(b_3.a')(b_1.(b'_1.c))(b_2.b'_0)]b_0 \\
&= a[(b_1.a')(b_2.(b'_1.c))(b_3.b'_0)]b_0 \\
&= a[b_1.(a'(b'_1.c)b'_0)]b_0 \\
&= F(a\#\bar{b})(a'(b'_1.c)b'_0) \\
&= F(a\#\bar{b})(F(a'\#\bar{b}')(c)) \\
&= F(a\#\bar{b})F(a'\#\bar{b}')(c).
\end{aligned}$$

We deduce that $F[(a\#\bar{b})(a'\#\bar{b}')] = F(a\#\bar{b})F(a'\#\bar{b}')$. Therefore F is a dyslectic (S,H)-dimodule algebra homomorphism. We do the same work for the map G, since $\overline{End_S(A)} \cong {}_SEnd(A)$ and we use the H-action and H-coaction defined in Lemma 2.2.3.

□

Let A be a faithfully projective dyslectic (S,H)-dimodule algebra. We say that A is a *dyslectic* (S,H)*-dimodule Azumaya algebra*, that is, an Azumaya algebra in the category $\mathscr{D}ys$-${}_S\mathscr{D}^H$, if A is faithfully projective, and the dyslectic (S,H)-dimodule algebra homomorphisms $F : A\#_S\bar{A} \to End_S(A)$ and $G : \bar{A}\#_S A \to \overline{End_S(A)}$ are isomorphisms.

An R-module is an R-progenerator if it is finitely generated projective and faithful (that is a generator) in the category of R-modules. Let A be a dyslectic (S,H)-dimodule algebra. Since H is cocommutative, if the coaction of H is trivial, then S is commutative, $\bar{A} = A^{op}$, and A is just an S-progenerator (S,H)-algebra for which the natural map $A \otimes_S A^{op} \to End_S(A)$ is an isomorphism of (S,H)-algebras. So A is an (S,H)-Azumaya algebra in the sense of [31]. Since H is commutative, if the action of H is trivial, then S is commutative, $\bar{A} = A^{op}$, and A is just an S-progenerator (S,H)-Hopf algebra for which the natural map $A \otimes_S A^{op} \to End_S(A)$ is an isomorphism of (S,H)-Hopf algebras. So A is an (S,H)-Hopf Azumaya algebra in the sense of [31].

Theorem 2.5.2. *The following properties hold:*

(i) *If M is a faithfully projective dyslectic (S,H)-dimodule, then $End_S(M)$ is a dyslectic (S,H)-dimodule Azumaya algebra.*

(ii) *If A and B are dyslectic (S,H)-dimodules Azumaya algebras, then $A\#_S B$ is a dyslectic (S,H)-dimodule Azumaya algebra.*

(iii) *If A is a dyslectic (S,H)-dimodule Azumaya algebra, then $\bar{A}$ is a dyslectic (S,H)-dimodule Azumaya algebra.*

Proof.

(i) It is clear that $End_S(M)$ is faithfully projective. The maps

$$F : End_S(M)\#_S\overline{End_S(M)} \to End_S(End_S(M))$$

$$\text{and } G : \overline{End_S(M)}\#_S End_S(M) \to \overline{End_S(End_S(M))}$$

are isomorphisms since, from Proposition 2.4.4, we have

$$\overline{End_S(M)} \cong {}_S End(M) \cong End_S(M^\star)$$

as dyslectic (S,H)-dimodule algebras. So we have

$$\begin{aligned} End_S(M)\#_S\overline{End_S(M)} &\cong End_S(M)\#_S End_S(M^\star) & \\ &\cong End_S(M\tilde{\otimes}_S M^\star) & \text{Proposition 2.4.5} \\ &\cong End_S(End_S(M)) & \text{Lemma 2.4.6}(ii) \end{aligned}$$

and

$$\begin{aligned} \overline{End_S(M)}\#_S End_S(M) &\cong End_S(M^\star)\#_S End_S(M) \\ &\cong End_S(M^\star\tilde{\otimes}_S M) \\ &\cong End_S((M\tilde{\otimes}_S M^\star)^\star) \\ &\cong End_S((End_S(M))^\star) \\ &\cong \overline{End_S(End_S(M))} \end{aligned}$$

(ii) Since $A\tilde{\otimes}_S B$ is faithfully projective so is $A\#_S B$. Using Propositions 2.4.3, 2.4.5, and 2.4.8, the maps

$$F : (A\#_S B)\#_S\overline{A\#_S B} \to End_S(A\#_S B) \quad \text{and} \quad G : \overline{A\#_S B}\#_S(A\#_S B) \to \overline{End_S(A\#_S B)}$$

are dyslectic (S,H)-dimodule algebra isomorphisms:

$$\begin{aligned} (A\#_S B)\#_S\overline{A\#_S B} &\cong A\#_S B\#_S\bar{B}\#_S\bar{A} \\ &\cong A\#_S End_S(B)\#_S\bar{A} \\ &\cong A\#_S\bar{A}\#_S End_S(B) \\ &\cong End_S(A)\#_S End_S(B) \\ &\cong End_S(A\tilde{\otimes}_S B) \\ &\cong End_S(A\#_S B) \end{aligned}$$

and

$$\begin{aligned}
\overline{A\#_S B}\#_S(A\#_S B) &\cong \bar{B}\#_S\bar{A}\#_S A\#_S B\\
&\cong \bar{B}\#_S\overline{End_S(A)}\#_S B\\
&\cong \bar{B}\#_S B\#_S\overline{End_S(A)}\\
&\cong End_S(B^\star)\#_S End_S(A^\star)\\
&\cong End_S(B^\star\tilde{\otimes}_S A^\star)\\
&\cong End_S((A\tilde{\otimes}_S B)^\star)\\
&\cong \overline{End_S(A\tilde{\otimes}_S B)}\\
&\cong \overline{End_S(A\#_S B)}.
\end{aligned}$$

So F and G are isomorphisms.

(iii) Since A is faithfully projective so is $\bar{A}$. Using Propositions 2.4.4 and 2.4.3, the map $F : \bar{A}\#_S\bar{\bar{A}} \to End_S(\bar{A})$ and $G : \bar{\bar{A}}\#_S\bar{A} \to \overline{End_S(\bar{A})}$ are dyslectic (S,H)-dimodule algebra isomorphisms. We have:

$$\bar{A}\#_S\bar{\bar{A}} \cong \overline{\bar{A}\#_S A} \cong \overline{\overline{End_S(A)}} \cong End_S(A) \cong End_S(\bar{A}),$$

$$\text{and} \qquad \bar{\bar{A}}\#_S\bar{A} \cong \overline{A\#_S\bar{A}} \cong \overline{End_S(A)} \cong \overline{End_S(\bar{A})}$$

so F and G are isomorphisms.

□

We will say that a dyslectic (S,H)-dimodule Azumaya algebra E is trivial if $E \cong End_S(P)$ as dyslectic (S,H)-dimodule algebras, for some faithfully projective dyslectic (S,H)-dimodule P. If a dyslectic (S,H)-dimodule Azumaya algebra E is trivial, then so are $E^\star$ and $\bar{E}$. If M and N are faithfully projective dyslectic (S,H)-dimodules, then so is $M\tilde{\otimes}_S N$. It follows from Proposition 2.4.5 and Theorem 2.5.2 that the braided product of two trivial dyslectic (S,H)-dimodule Azumaya algebras is a trivial dyslectic (S,H)-dimodule Azumaya algebra. When A is a dyslectic (S,H)-dimodule Azumaya algebra, then we have that $A\#_S\bar{A}$ and $End_S(A)$ are isomorphic as dyslectic (S,H)-dimodule Azumaya algebras, and $\bar{A}\#_S A$ and $\overline{End_S(A)}$ are isomorphic as dyslectic (S,H)-dimodule Azumaya algebras.

We will say that two dyslectic (S,H)-dimodule Azumaya algebras A and B are equivalent if there exist trivial dyslectic (S,H)-dimodule Azumaya algebras E_1 and E_2 such that $A\#_S E_1 \cong B\#_S E_2$ as dyslectic (S,H)-dimodule Azumaya algebras.

Lemma 2.5.3. *The above relation is an equivalence relation on the collection of dyslectic (S,H)-dimodule Azumaya algebras.*

Proof. The only thing we have to show is transitivity. Suppose A,B and C are dyslectic (S,H)-dimodule Azumaya algebras for which A is equivalent to B and B is equivalent to C. Then there exist faithfully projective dyslectic (S,H)-dimodules E_1,E_2,E_3 and E_4 such that $A\#_S End_S(E_1) \cong B\#_S End_S(E_2)$ and $B\#_S End_S(E_3) \cong C\#_S End_S(E_4)$ as dyslectic (S,H)-dimodule Azumaya algebras. We have the following dyslectic (S,H)-dimodule Azumaya algebras isomorphisms:

$$\begin{aligned} A\#_S End_S(E_1\tilde{\otimes}_S E_3) &\cong A\#_S End_S(E_1)\#_S End_S(E_3)\\ &\cong B\#_S End_S(E_2)\#_S End_S(E_3)\\ &\cong B\#_S End_S(E_3)\#_S End_S(E_2)\\ &\cong C\#_S End_S(E_4)\#_S End_S(E_2)\\ &\cong C\#_S End_S(E_4\tilde{\otimes}_S E_2), \end{aligned}$$

so this relation is transitive, then it is an equivalence relation.

□

We have now collected all of the ingredients necessary to define the Brauer group for the braided monoidal category $\mathscr{D}ys\text{-}_S\mathscr{D}^H$.

Definition 2.5.4. *The Brauer-Clifford-Long group for the category of dyslectic (S,H)-dimodules Azumaya algebras is the set $BD(S,H)$ of equivalence classes of dyslectic (S,H)-dimodule Azumaya algebras modulo the relation defined by taking $\#_S$-products with trivial dyslectic (S,H)-dimodule Azumaya algebras.*

We remind the reader that our Azumaya algebras in $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ are assumed to be left and right faithfully projective. From the viewpoint of [23], these algebras constitute a closed braided monoidal category and so the Brauer-Clifford-Long group we have described is the Brauer group of this category.

Theorem 2.5.5. *Let H be a commutative and cocommutative Hopf algebra and S an H-commutative H-dimodule algebra. Then $BD(S,H)$ is a group. If $[A]$ and $[B]$ denote the equivalence classes of dyslectic (S,H)-dimodules Azumaya algebras A and B, then in $BD(S,H)$, we will have $[A].[B]=[A\#_S B]$. The identity of $BD(S,H)$ is the equivalence class $[S]$ consisting of all trivial dyslectic (S,H)-dimodules Azumaya algebras, and $[A]^{-1}=\left[\bar{A}\right]$, for all $[A]\in BD(S,H)$.*

Proof. The product in $BD(S,H)$ is well defined by Propositions 2.4.3, 2.4.5, 2.4.8, and Theorem 2.5.2(ii). This product is associative and has an identity element (Proposition 2.4.3).

Let A be a dyslectic (S,H)-dimodule Azumaya algebra. By Theorem 2.5.2(iii), $\bar{A}$ is a dyslectic (S,H)-dimodule Azumaya algebra. It is clear that $\left[\bar{A}\right]$ is the inverse of $[A]$ in $BD(S,H)$. Therefore $BD(S,H)$ is a group.

□

Example 2.5.6.

We consider again Example 2.1.1.
A G-graded algebra S is an R-algebra S which is a G-graded R-module $S=\bigoplus_{\sigma\in G}S_\sigma$ such that $ss'\in S_{\sigma\sigma'}$ for all $s\in S_\sigma$ and $s'\in S_{\sigma'}$. An R-module is a G-graded algebra if and only if it is an RG-comodule algebra. A left G-module algebra is exactly a left RG-module algebra. If S is a G-module algebra, we will denote the smash product of S with RG by $S\#G$. A G-dimodule algebra is a G-graded algebra which is a G-dimodule and a G-module algebra. Let S be a G-dimodule

algebra. Following the terminology of [15], an (S,G)-dimodule will be a G-graded S-module which is an $S\#G$-module and a G-dimodule. S is graded-commutative if $ss' = (\sigma.s')s$ for all $s \in S_\sigma, s' \in S$. Then S is graded-commutative if and only if it is RG-commutative. Let S be a graded-commutative dimodule algebra. We denote by ${}_S\mathscr{D}^G$ the category of (S,G)-dimodules: its morphisms are the graded S-linear maps of degree e which are G-linear. An (S,G)-dimodule M is dyslectic if $s \rightharpoonup m = (\sigma'.s)(\sigma m)$ for all $s \in S_\sigma$ and $m \in M_{\sigma'}$. We denote by $Dys\text{-}{}_S\mathscr{D}^G$ the category of dyslectic (S,G)-dimodules: it is an abelian full subcategory of ${}_S\mathscr{D}^G$. If M and N are dyslectic (S,G)-dimodules, so is $M \otimes_S N$. We can show that $(Dys\text{-}{}_S\mathscr{D}^G, \otimes_S, S, \gamma)$ is a braided monoidal category, where $\gamma_{M,N}(m \otimes_S n) = \sigma n \otimes_S m$ for $m \in M_\sigma$ and $n \in N$. Then we can define a Brauer group in $Dys\text{-}{}_S\mathscr{D}^G$ which we will denote by $BD(S,G)$: it is a generalization of the Brauer-Long group $BD(R,G)$ of R with respect to G introduced in [15]. We can show that the categories $Dys\text{-}{}_S\mathscr{D}^{RG}$ and $Dys\text{-}{}_S\mathscr{D}^G$ are equivalent and that the Brauer-Clifford-Long group $BD(S,RG)$ is isomorphic to the Brauer group $BD(S,G)$.

Before ending our example, let's say a few words about RG-dimodules in the case of a cyclic group of order 2, $G = \{e,\sigma\}$. Set $H = RG$, the left group algebra of G over R. If S is an H-dimodule algebra, then for any $s \in S$, we have

$$\rho(s) = s_{(0)} \otimes e + s_{(1)} \otimes \sigma \tag{2.5.1}$$

Therefore for any $s,t \in S$, we have:

(i) $s = s_{(0)} + s_{(1)}$ with $s_{(0)}, s_{(1)} \in S$,

(ii) $(st)_{(0)} = s_{(0)}t_{(0)} + s_{(1)}t_{(1)}$,

(iii) $(st)_{(1)} = s_{(0)}t_{(1)} + s_{(1)}t_{(0)}$,

(iv) $(\sigma s)_{(0)} = \sigma s_{(0)}$ and $(\sigma s)_{(1)} = \sigma s_{(1)}$.

The RG-commutativity of S means

$$ss' = s's_{(0)} + (\sigma.s')s_{(1)} \tag{2.5.2}$$

For an (S,RG)-dimodule M, where S is an RG-commutative RG-dimodule algebra, the dyslectic condition means

$$s \rightharpoonup m = [s_{(0)} \rightharpoonup m_{(0)}] + [s_{(1)} \rightharpoonup (\sigma m_{(0)})] + [(\sigma.s_{(0)}) \rightharpoonup m_{(1)}] + [(\sigma.s_{(1)}) \rightharpoonup (\sigma m_{(1)})].$$

Example 2.5.7.

We keep the notations of Example 2.1.2.

If M is an L-comodule, we denote by $m \mapsto m_{(0)} \otimes m_{(1)}$ its comodule structure. Let S be a K-dimodule algebra and T be an L-dimodule algebra. Then $S \times T$ is an H-dimodule algebra :

$$(\lambda, \lambda')(s,t) = (\lambda s, \lambda' t),$$

$$(k,l).(s,t) = (k.s, l.t)$$

and

$$(s,t)_{[0]} \otimes_R (s,t)_{[1]} = (s_0, t_{(0)}) \otimes_R (s_1, t_{(1)})$$

for $\lambda, \lambda' \in \Bbbk$, $k \in K, l \in L, s \in S$ and $t \in T$. If furthermore, S is K-commutative and T is L-commutative, then $S \times T$ is H-commutative. Let S be a K-commutative K-dimodule algebra and T an L-commutative L-dimodule algebra. Thus we can consider the categories ${}_S\mathscr{D}^K$, $Dys\text{-}{}_S\mathscr{D}^K$, ${}_T\mathscr{D}^L$, $Dys\text{-}{}_T\mathscr{D}^L$, ${}_{(S\times T)}\mathscr{D}^H$, $Dys\text{-}{}_{(S\times T)}\mathscr{D}^H$. We have three Brauer groups $BD(S,K)$, $BD(T,L)$ and $BD(S \times T, H)$. We want to establish some relations between these Brauer groups. Every object M of the category ${}_S\mathscr{D}^K$ (resp. $Dys\text{-}{}_S\mathscr{D}^K$) is an object of ${}_{S\times T}\mathscr{D}^H$ (resp. $Dys\text{-}{}_{(S\times T)}\mathscr{D}^H$) : $(s,t)m = sm$ (the first projection $S \times T \to S$), $(k,l).m = km$ (the first projection $K \times L \to K$) and $m_{[0]} \otimes m_{[1]} = m_0 \otimes (m_1, 0_L)$ for $s \in S$, $t \in T$ and $m \in M$. Every object M of the category ${}_T\mathscr{D}^L$ (resp. $Dys\text{-}{}_T\mathscr{D}^L$) is an object of ${}_{(S\times T)}\mathscr{D}^H$ (resp. $Dys\text{-}{}_{(S\times T)}\mathscr{D}^H$) : $(s,t)m = tm$ (the second projection $S \times T \to T$), $(k,l).m = lm$ (the second projection $K \times L \to L$) and $m_{[0]} \otimes m_{[1]} = m_{(0)} \otimes (0_K, m_{(1)})$.

For every M in ${}_S\mathscr{D}^K$ or in ${}_T\mathscr{D}^L$, we have $End_S(M) = End_{S\times T}(M)$.

Let M and N be two $(S \times T, H)$-dimodules. Then the braiding map in ${}_{(S\times T)}\mathscr{D}^H$ is defined by

$$\gamma_{M,N}(m \otimes_{S\times T} n) = (m_{[1]}.n) \otimes_{S\times T} m_{[0]} \quad \forall \quad m \in M, n \in N.$$

When M and N are objects of ${}_S\mathscr{D}^K$ viewed as objects of ${}_{(S\times T)}\mathscr{D}^H$, we have

$$\gamma_{M,N}(m \otimes_{S\times T} n) = (m_1.n) \otimes_{S\times T} m_0$$

(note the similarity with the braiding of ${}_S\mathscr{D}^K$).

When M and N are objects of ${}_T\mathscr{D}^L$ viewed as objects of ${}_{(S\times T)}\mathscr{D}^H$, we have

$$\gamma_{M,N}(m \otimes_{S\times T} n) = (m_{(1)}.n) \otimes_{S\times T} m_{(0)}$$

(note the similarity with the braiding of ${}_T\mathscr{D}^L$). Now when M (resp. N) is an object of ${}_S\mathscr{D}^K$ (resp. an object of ${}_T\mathscr{D}^L$) viewed as an object of ${}_{(S\times T)}\mathscr{D}^H$, we will consider the trivial braiding, that is,

$$\gamma_{M,N}(m \otimes_{S\times T} n) = n \otimes_{S\times T} m.$$

In the same way, when M (resp. N) is an object of ${}_T\mathscr{D}^L$ (resp. an object of ${}_S\mathscr{D}^K$) viewed as an object of ${}_{(S\times T)}\mathscr{D}^H$, we will consider the trivial braiding.

Every faithfully projective object in $Dys\text{-}{}_S\mathscr{D}^K$ is faithfully projective in $Dys\text{-}{}_{(S\times T)}\mathscr{D}^H$. Likewise, every faithfully projective object in $Dys\text{-}{}_T\mathscr{D}^L$ is faithfully projective in $Dys\text{-}{}_{(S\times T)}\mathscr{D}^H$. Let A be a dyslectic (S,K)-dimodule algebra (resp. a dyslectic (T,L)-dimodule algebra), then A is a dyslectic $(S \times T, H)$-dimodule algebra. If $\bar{A}$ is the K-opposite algebra of A as a dyslectic (S,K)-dimodule algebra, then $\bar{A}$ is the H-opposite of A as a dyslectic $(S \times T, H)$-dimodule algebra. Likewise, if $\bar{A}$ is the L-opposite algebra of A as a dyslectic (T,L)-dimodule algebra, then $\bar{A}$ is the H-opposite of A as a dyslectic $(S \times T, H)$-dimodule algebra. Dyslectic (S,K)-dimodule Azumaya algebras and dyslectic (T,L)-dimodule Azumaya algebras are dyslectic $(S \times T, H)$-dimodule Azumaya algebras.

If A and B are dyslectic (S,K)-dimodule algebras, the braided product of A and B in $Dys\text{-}_{(S\times T)}\mathscr{D}^H$ is given by

$$(a\#_{S\times T}b)(a'\#_{S\times T}b') = (a(b_1.a'))\#_{S\times T}(b_0b'); \quad a,a' \in A, b,b' \in B.$$

If A and B are dyslectic (T,L)-dimodule algebras, the braided product of A and B in $Dys\text{-}_{(S\times T)}\mathscr{D}^H$ is given by

$$(a\#_{S\times T}b)(a'\#_{S\times T}b') = (a(b_{(1)}.a'))\#_{S\times T}(b_{(0)}b'); \quad a,a' \in A, b,b' \in B.$$

We have injective group homomorphisms

$$BD(S,K) \hookrightarrow BD(S\times T,H)$$

and

$$BD(T,L) \hookrightarrow BD(S\times T,H).$$

If $T=\mathbb{k}$ as a trivial L-dimodule algebra, we deduce an injective group homomorphism

$$BD(S,K) \hookrightarrow BD(S,H).$$

If $S=\mathbb{k}$ as a trivial K-dimodule algebra, we deduce an injective group homomorphism

$$BD(T,L) \hookrightarrow BD(T,H).$$

Let A (resp. B) be a dyslectic (S,K)-dimodule algebra (resp. a dyslectic (T,L)-dimodule algebra). Then the braided product of A and B in $Dys\text{-}_{(S\times T)}\mathscr{D}^H$ is the trivial one, that is,

$$(a\#_{S\times T}b)(a'\#_{S\times T}b') = (aa')\otimes_{S\times T}(bb'); \quad a,a' \in A, b,b' \in B.$$

It follows that the classes of A and B in $BD(S\times T,H)$ commute. Furthermore if A (resp. B) is a dyslectic (S,K)-dimodule Azumaya algebra (resp. a dyslectic (T,L)-dimodule Azumaya algebra), then $A\#_{S\times T}B$ is a dyslectic (S,T)-dimodule Azumaya algebra. We have a well-defined injective group homomorphism

$$BD(S,K)\times BD(T,L) \longrightarrow BD(S\times T,H); \ ([A],[B]) \longmapsto [A\otimes_{S\times T}B].$$

We can show that the intersection of $BD(S,K)$ and $BD(T,L)$ in $BD(S\times T,H)$ is trivial.

2.6 Elementary homomorphisms between Brauer-Clifford-Long groups

We are going to present some elementary homomorphisms between Brauer-Clifford-Long groups that are induced by scalar extensions and central twists.

We first consider scalar extensions. Let R' be a commutative ring with trivial H-action and H-coaction. Fix a ring homomorphism from R to R'. Then $H' = R' \otimes H$ equipped with its natural R'-module structure is a Hopf algebra over R'. Let S be an H-commutative H-dimodule algebra. Then $R' \otimes S$ is an H'-commutative H'-dimodule algebra. Let M be an (S,H)-dimodule. Then $R' \otimes M$ equipped with its natural $R' \otimes S$-module structure is an $(R' \otimes S, H')$-dimodule. If M is dyslectic then so is $R' \otimes M$. If M is faithfully projective as an S-module, then $R' \otimes M$ is faithfully projective as an $R' \otimes S$-module. Furthermore, if A is an (S,H)-dimodule (Azumaya) algebra, then $R' \otimes A$ will be an $(R' \otimes S, H')$-dimodule (Azumaya) algebra, and $\overline{R' \otimes A} \cong R \otimes \bar{A}$. The canonical nature of these identifications allows us to lift this to a homomorphism between the Brauer-Clifford groups.

Proposition 2.6.1. *Let S be an H-commutative H-dimodule algebra. Suppose that R' is a commutative ring with trivial H-action and H-coaction and there is a homomorphism ring from R' to R. Then the map $BD(S,H) \to BD(R' \otimes S, R' \otimes H)$ given by $[A] \mapsto [R' \otimes A]$, for all (S,H)-dimodule Azumaya algebras A, is a group homomorphism.*

Central twists also induce homomorphisms between Brauer-Clifford-Long groups. Let S be an H-commutative H-dimodule algebra. Let $H\text{-}Aut_R(S)$ be the group of H-dimodule algebra automorphisms of S. We claim there is an action of $H\text{-}Aut_R(S)$ on the Brauer-Clifford-Long group. For $M \in {}_S\mathscr{D}^H$ and τ in $H\text{-}Aut_R(S)$, let ${}_\tau M$ be equal to M as an H-dimodule, but has left S-module structure given by

$$s \triangleright m = \tau^{-1}(s) \rightharpoonup m$$

for all $s \in S, m \in M$. Using the H-linearity and the colinearity of τ, we can see that ${}_\tau M \in {}_S\mathscr{D}^H$. The corresponding right S-module structure on ${}_\tau M$ is given by

$$m \triangleleft s = m \leftharpoonup \tau^{-1}(s).$$

Using the H-linearity and the H-colinearity of τ, we can show that if M is an object of $Dys\text{-}{}_S\mathscr{D}^H$ then so is ${}_\tau M$.

Lemma 2.6.2. *Let S be an H-commutative H-dimodule algebra. Let $\tau \in H\text{-}Aut_R(S)$. Let $M, N \in {}_S\mathscr{D}^H$. Then the following hold.*

(i) ${}_\tau(M \tilde{\otimes}_S N) = {}_\tau M \tilde{\otimes}_S {}_\tau N$;

(ii) *M is finitely generated projective as a right (left) S-module if and only if ${}_\tau M$ is finitely generated projective as a right (left) S-module;*

(iii) *If M is finitely generated projective as a right S-module, then ${}_\tau Hom_S(M;N)$ and $Hom_S({}_\tau M, {}_\tau N)$ are isomorphic in ${}_S\mathscr{D}^H$,*

(iv) *If M is finitely generated projective as a left S-module, then $_\tau({}_SHom(M,N))$ and $_SHom(_\tau M, _\tau N)$ are isomorphic in $_S\mathscr{D}^H$; and*

(v) *M is S-faithfully projective in Dys-$_S\mathscr{D}^H$ if and only if $_\tau M$ is S-faithfully projective in Dys-$_S\mathscr{D}^H$.*

Proof.

(i) The new S-action is well defined. The identity map $Id : {}_\tau(M\tilde{\otimes}_S N) \to {}_\tau M\tilde{\otimes}_S {}_\tau N$ is H-linear, H-colinear and S-linear. So $_\tau(M\tilde{\otimes}_S N) = {}_\tau M\tilde{\otimes}_S {}_\tau N$ in $_S\mathscr{D}^H$.

(ii) Since M is finitely generated projective, $m = \sum_{i\in I} f^{(i)}(m) \rightharpoonup m^{(i)}$, where $\{m^{(i)}, f^{(i)}\}$ is a dual basis of the left S-module M. For all $s \in S, m \in M$; we have

$$s \triangleright m = \tau^{-1}(s) \rightharpoonup m \iff \tau(s) \triangleright m = \tau^{-1}(\tau(s)) \rightharpoonup m = s \rightharpoonup m.$$

We deduce that, for every $m \in M$,

$$m = \sum_{i\in I} f^{(i)}(m) \rightharpoonup m^{(i)} = \sum_{i\in I} \tau(f^{(i)}(m)) \triangleright m^{(i)} = \sum_{i\in I} (\tau \circ f^{(i)})(m) \triangleright m^{(i)},$$

that is, M is finitely generated projective as a left S-module iff $_\tau M$ is finitely generated projective as a left S-module.

(iii) Let M,N in $_S\mathscr{D}^H$ and $f \in Hom_S(M,N)$.
By definition, M (resp. N) is equal to $_\tau M$ (resp. $_\tau N$) as objects of $\mathscr{D}^H$ then, f is right S-linear, left H-linear and right H-colinear from M to N, if and only if it is so from $_\tau M$ to $_\tau N$ and $s \triangleright f = s \rightharpoonup f$. On the other hand, $_\tau Hom_S(M,N)$ is equal to $Hom_S(_\tau M, _\tau N)$ as objects of $\mathscr{D}^H$ and the identity map from $_\tau Hom_S(M,N)$ to $Hom_S(_\tau M, _\tau N)$ is S-linear.
Therefore $_\tau Hom_S(M,N) \cong Hom_S(_\tau M, _\tau N)$ as objects of $_S\mathscr{D}^H$.

(iv) Adapt the proof of (iii).

(v) By definition, $M \in Dys$-$_S\mathscr{D}^H \iff {}_\tau M \in Dys$-$_S\mathscr{D}^H$, and it is clear that M is faithfully projective as a left S-module iff $_\tau M$ faithfully projective as a left S-module.

□

Definition 2.6.3. *Let S be an H-commutative H-dimodule algebra. Let A be an algebra in Dys-$_S\mathscr{D}^H$. For any $\tau \in H$-$Aut_R(S)$, we define $_\tau A$ to be equal to A as an H-dimodule algebra, but equal to $_\tau A$ as an S-module.*

Lemma 2.6.4. *Let S be an H-commutative H-dimodule algebra. Let $\tau \in H$-$Aut_R(S)$ and A be an algebra in Dys-$_S\mathscr{D}^H$. Then $_\tau A$ is an algebra in Dys-$_S\mathscr{D}^H$.*

Proof. By Lemma 2.6.2 (i), ${}_\tau(A\tilde{\otimes}_S A) = {}_\tau A\tilde{\otimes}_{S\,\tau}A$ in $Dys\text{-}{}_S\mathscr{D}^H$. The product of A induces a well-defined product on ${}_\tau A$ (see Lemma 2.6.2 (i)). The unit map on A induces a well-defined unit map on ${}_\tau A$. This product and this unit map satisfy the usual properties of associative algebra. Therefore ${}_\tau A$ is a dyslectic (S,H)-dimodule algebra, that is, it is an algebra in $Dys\text{-}{}_S\mathscr{D}^H$.

□

Lemma 2.6.5. *Let S be an H-commutative H-dimodule algebra. Let $\tau \in H\text{-}Aut_R(S)$. Then the following hold.*

(i) *If M is faithfully projective as an S-module in $Dys\text{-}{}_S\mathscr{D}^H$, then ${}_\tau End_S(M) \cong End_S({}_\tau M)$ and ${}_\tau({}_S End(M)) \cong {}_S End({}_\tau M)$ as algebras in $Dys\text{-}{}_S\mathscr{D}^H$;*

(ii) *if A is an algebra in $Dys\text{-}{}_S\mathscr{D}^H$, then ${}_\tau A$ is an algebra in $Dys\text{-}{}_S\mathscr{D}^H$, and $\overline{{}_\tau A} = {}_\tau\bar{A}$ as algebras in $Dys\text{-}{}_S\mathscr{D}^H$;*

(iii) *if A and B are algebras in $Dys\text{-}{}_S\mathscr{D}^H$, then ${}_\tau(A\#_S B)$ is an algebra in $Dys\text{-}{}_S\mathscr{D}^H$ and ${}_\tau(A\#_S B) \cong {}_\tau A\#_{S\,\tau}B$ as algebras in $Dys\text{-}{}_S\mathscr{D}^H$; and*

(iv) *if A is an Azumaya algebra in $Dys\text{-}{}_S\mathscr{D}^H$, then so is ${}_\tau A$.*

Proof.

(i) From Lemma 2.6.2 (iii) and (iv), ${}_\tau End_S(M) \cong End_S({}_\tau M)$ and ${}_\tau({}_S End(M)) \cong {}_S End({}_\tau M)$ in ${}_S\mathscr{D}^H$. Since $End_S(M)$ and ${}_S End(M)$ are algebras in $Dys\text{-}{}_S\mathscr{D}^H$, so are ${}_\tau End_S(M)$ and ${}_\tau({}_S End(M))$ (Lemma 2.6.4). Clearly $End_S({}_\tau M)$ and ${}_S End({}_\tau M)$ are algebras in $Dys\text{-}{}_S\mathscr{D}^H$. Therefore ${}_\tau End_S(M) \cong End_S({}_\tau M)$ and ${}_\tau({}_S End(M)) \cong {}_S End({}_\tau M)$ as algebras in $Dys\text{-}{}_S\mathscr{D}^H$.

(ii) Since ${}_\tau A$ and $\bar{A}$ are algebras in $Dys\text{-}{}_S\mathscr{D}^H$, so are $\overline{{}_\tau A}$ and ${}_\tau\bar{A}$ (Lemmas 2.4.2 and 2.6.4) and we have:

$$\overline{s \triangleright a} = \overline{\tau^{-1}(s) \rightharpoonup a} = \tau^{-1}(s) \rightharpoonup \bar{a} = s \triangleright \bar{a}.$$

Therefore $\overline{{}_\tau A}$ is equal to ${}_\tau\bar{A}$ as algebras in $Dys\text{-}{}_S\mathscr{D}^H$.

(iii) Let A and B be algebras in $Dys\text{-}{}_S\mathscr{D}^H$. $A\#_S B$ is an algebra in $Dys\text{-}{}_S\mathscr{D}^H$(Proposition 2.4.3 (i)), so by Lemma 2.6.4 , ${}_\tau(A\#_S B)$ is an algebra in $Dys\text{-}{}_S\mathscr{D}^H$. Since ${}_\tau A$ and ${}_\tau B$ are algebras in $Dys\text{-}{}_S\mathscr{D}^H$ (Lemma 2.6.4) , then so is ${}_\tau A\#_{S\,\tau}B$. We also have ${}_\tau(A\#_S B) = {}_\tau A\#_{S\,\tau}B$ in ${}_S\mathscr{D}^H$, therefore ${}_\tau(A\#_S B) \cong {}_\tau A\#_{S\,\tau}B$ as algebras in $Dys\text{-}{}_S\mathscr{D}^H$.

(iv) Let A be an Azumaya algebra in $Dys\text{-}_S\mathscr{D}^H$. Using the definition of an Azumaya algebra and the fact that $\overline{_\tau A} =_\tau \bar{A}$, we get the assertion.

$\square$

Proposition 2.6.6. *H-$Aut_R(S)$ acts by automorphisms on $BD(S,H)$. The action is given by $\tau.[A] = [_\tau A]$, for any Azumaya algebra A in $Dys\text{-}_S\mathscr{D}^H$ and $\tau \in H\text{-}Aut_R(S)$.*

Proof. Let A and B be two equivalent Azumaya algebras in $Dys\text{-}_S\mathscr{D}^H$. Then there exist faitfully projective (S,H)-dimodules M and N such that $A\#_S End_S(M) \cong B\#_S End_S(N)$ as dimodules algebras. By Lemma 2.6.5(i) and (iii), we have:

$$\begin{aligned} {_\tau A}\#_S\, End_S(_\tau M) &\cong {_\tau A}\#_S\, {_\tau End_S(M)} \\ &\cong {_\tau(A\#_S\, End_S(M))} \\ &\cong {_\tau(B\#_S\, End_S(N))} \\ &\cong {_\tau B}\#_S\, {_\tau End_S(N)} \\ &\cong {_\tau B}\#_S\, End_S(_\tau N). \end{aligned}$$

Then $_\tau A$ and $_\tau B$ are Brauer equivalent Azumaya algebras in $Dys\text{-}_S\mathscr{D}^H$. So the action of $H\text{-}Aut_R(S)$ on $BD(S,H)$ is well defined.

By Lemma 2.6.5 (iii), $_\tau(A\#_S B) \cong {_\tau A}\#_S {_\tau B}$ as algebras in $Dys\text{-}_S\mathscr{D}^H$. Then, we have

$$\tau.([A][B]) = \tau.([A\#_S B]) = [_\tau(A\#_S B)] = [_\tau A\#_S {_\tau B}] = [_\tau A][_\tau B] = (\tau.[A])(\tau.[B]),$$

so the action of τ on $BD(S,H)$ is a homomorphism of groups.

Let A and B be Azumaya algebras in $Dys\text{-}_S\mathscr{D}^H$. If $\tau,\tau' \in H\text{-}Aut_R(S)$, $_{\tau\tau'}A$ and $_\tau(_{\tau'}A)$ are equal to A as dimodules. The map $\phi : {_{\tau\tau'}A} \to {_\tau(_{\tau'}A)}$ given by $\phi(a) = a$, is an isomorphism in $Dys\text{-}_S\mathscr{D}^H$. So we have

$$(\tau\tau').[A] = \left[_{(\tau\tau')}A\right] = [_\tau(_{\tau'}A)] = \tau.([_{\tau'}A]) = \tau.(\tau'.[A]).$$

We have $Id.[A] = [A]$, where Id is the identity element of $H\text{-}Aut_R(S)$ and note that $\tau.[S] = [S]$, $\forall \tau \in H\text{-}Aut_R(S)$. Therefore $H\text{-}Aut_R(S)$ acts by automorphism on $BD(S,H)$.

$\square$

We conclude our paper by establishing an anti-isomorphism of groups between our Brauer-Clifford-Long group $BD(S,H)$ and the Brauer-Clifford-Long group studied in [32]. We give detailed proof for a best understanding of this isomorphism.

2.7 Anti-isomorphism between $BD(S,H)$ and $BQ(S^{op},H)$

Let H be a Hopf algebra with a bijective antipode. For Hopf Yetter-Drinfel'd H-modules M and N, there exist a Yetter-Drinfel'd H- module isomorphism $\gamma'_{M,N}$ from $M\tilde{\otimes}N$ to $N\tilde{\otimes}M$ defined by

$$\gamma'_{M,N}(m\tilde{\otimes}n) = n_0\tilde{\otimes}n_1 m, \quad \text{for all } m \in M, n \in N \tag{2.7.1}$$

with inverse

$$\gamma'^{-1}_{M,N}(n\tilde{\otimes}m) = S_H(n_1)m\tilde{\otimes}n_0, \quad \text{for all } m \in M, n \in N \tag{2.7.2}$$

According to [18, (1.2.4)] and [17, (1.4)], $(\mathcal{Q}^H, \tilde{\otimes}_k, \gamma'_{M,N}, k)$ is a braided monoidal category where $\mathcal{Q}^H$ is the category of Hopf Yetter Drinfel'd H-modules.
A Hopf Yetter Drinfel'd H-module algebra T is said to be H-commutative if:

$$tt' = t'_0(t'_1.t), \quad \text{for all } t, t' \in T. \tag{2.7.3}$$

Let H be commutative and cocommutative. Thus, the category of H-dimodules $\mathcal{D}^H$ and the category of Hopf Yetter Drinfel'd H-modules $\mathcal{Q}^H$ are equivalent.

Lemma 2.7.1. *An H-dimodule algebra S is H-commutative if and only if its opposite S^{op} is H-commutative as a Hopf Yetter-Drinfel'd H-module algebra.*

Using the H-commutativity of S^{op} with the braiding of Hopf Yetter-Drinfel'd H-modules, we can form the category of Hopf Yetter-Drinfel'd (S^{op},H)-modules ${}_{S^{op}}\mathcal{Q}^H$. According to [32], we have a Brauer-Clifford-Long group $BQ(S^{op},H)$ of dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-module Azumaya algebras.

Let M be an (S,H)-dimodule. Then M becomes a right S^{op}-module if we set:

$$m \blacktriangleleft s^o = s \rightharpoonup m, \text{ for all } m \in M, s \in S. \tag{2.7.4}$$

Since S^{op} is an H-commutative Hopf Yetter Drinfel'd H-module algebra, according to the relation (18) of [32], M becomes a left S^{op}-module:

$$s^o \blacktriangleright m = (S_H(s_1)m) \blacktriangleleft s_0^o, \text{ for all } m \in M, s \in S. \tag{2.7.5}$$

Using the relation (2.7.4), we obtain:

$$s^o \blacktriangleright m = s_0 \rightharpoonup (S_H(s_1)m), \text{ for all } m \in M, s \in S. \tag{2.7.6}$$

From now on, H is commutative and cocommutative and S is an H-commutative H-dimodule algebra.

Lemma 2.7.2. *Let M be an (S,H)-dimodule. Considering the actions defined above,*

(i) *M is an object of ${}_S\mathcal{D}^H$ iff it is an object of ${}_{S^{op}}\mathcal{Q}^H$,*

(ii) *M is a dyslectic object of ${}_S\mathcal{D}^H$ iff it is a dyslectic object of ${}_{S^{op}}\mathcal{Q}^H$.*

Proof.

(i) The proof uses formula (2.7.4), the commutativity of H and the dimodule condition.

(ii) Let $M \in Dys\text{-}_S\mathscr{D}^H$. For all $m \in M$ and $s \in S$, we have from (2.3.4): $m \leftharpoondown s = s_0 \rightharpoonup (S_H(s_0)m)$, then

$$s^o \blacktriangleright m = s_0 \rightharpoonup (S_H(s_1)m) = m \leftharpoondown s = (m_1.s) \rightharpoonup m_0 = m_0 \blacktriangleleft (m_1.s)^o = m_0 \blacktriangleleft (m_1.s^o),$$

that is $M \in Dys\text{-}_{S^{op}}\mathscr{Q}^H$. Let $M \in Dys\text{-}_{S^{op}}\mathscr{Q}^H$, for all $m \in M, s \in S$, we have:

$$s \rightharpoonup m = m \blacktriangleleft s^o = s_0^o \blacktriangleright (s_1 m) = (s_1 m)_0 \blacktriangleleft ((s_1 m)_1.s_0^o) = ((s_1 m)_1.s_0) \rightharpoonup (s_1 m)_0 = (s_1 m) \leftharpoondown s_0,$$

this is the condition (2.3.3), then $M \in Dys\text{-}_S\mathscr{D}^H$.

□

Consider a dyslectic (S,H)-dimodule algebra A. Then $A \in {}_{S^{op}}\mathscr{Q}^H$. The opposite A^{op} of A is equal to A as an object of ${}_S\mathscr{D}^H$ but equiped with the product $a^o b^o = (ba)^o$ for all $a, b \in A$.

Lemma 2.7.3. (i) *If A is a dyslectic (S,H)-dimodule algebra, then A^{op} is a dyslectic Hopf Yetter Drinfel'd (S^{op},H)-module algebra: the left action of S^{op} on A^{op} is defined by*

$$s^o \blacktriangleright a^o = (s^o \blacktriangleright a)^o, \quad \textit{for all } a \in A, s \in S.$$

(ii) *If A is a dyslectic Hopf Yetter Drinfel'd (S^{op},H)-module algebra, then A^{op} is a dyslectic (S,H)-dimodule algebra: the left action of S on A^{op} is defined by*

$$s \rightharpoonup a^o = (s \rightharpoonup a)^o, \quad \textit{for all } a \in A, s \in S.$$

Proof.

(i) For all $a, b \in A$, we have:

$$\begin{aligned} a^o(s^o \blacktriangleright b^o) &= a^o[s_0 \rightharpoonup (S_H(s_1).b)]^o \\ &= [(s_0 \rightharpoonup (S_H(s_1).b))a]^o \\ &= [(b \leftharpoondown s)a]^o \\ &= [b(s \rightharpoonup a)]^o \\ &= (s \rightharpoonup a)^o b^o \\ &= (a^o \blacktriangleleft s^o)b^o, \end{aligned}$$

that is, the multiplication in A^{op} is well-defined.

It is clear that the product in A^{op} is S^{op}-linear and compatible with the H-action and the H-coaction. In addition, for all $s \in S$ and $a \in A$; we have:

$$s^o \blacktriangleright a^o = (s_0 \rightharpoonup (S_H(s_1).a))^o = (a \leftharpoondown s)^o = ((a_1.s) \rightharpoonup a_0)^o = a_0^o \blacktriangleleft (a_1.s^o)$$

Therefore A^{op} is a dyslectic Hopf Yetter Drinfel'd (S^{op},H)-module algebra.

(ii) We use the same method as in (i).

□

For every algebra A in $Dys\text{-}_S\mathscr{D}^H$, we have $A^{opop} = A$ as algebras in $Dys\text{-}_S\mathscr{D}^H$. Similarly, for every algebra A in $Dys\text{-}_{S^{op}}\mathscr{Q}^H$, we have $A^{opop} = A$ as algebras in $Dys\text{-}_{S^{op}}\mathscr{Q}^H$.

To avoid all confusion, the product $\#_{S^{op}}$ will be denoted by $\natural$, that is for all $A,B \in Dys\text{-}_{S^{op}}\mathscr{Q}^H$ we have:

$$(a\natural b)(a'\natural b') = aa'_0\natural(a'_1.b)b' \quad \forall a,a' \in A, \quad b,b' \in B. \tag{2.7.7}$$

Lemma 2.7.4. *(i) If A and B are dyslectic (S,H)-dimodules algebras, then $(A\#_S B)^{op} \cong B^{op}\natural A^{op}$ as dyslectic Hopf Yetter- Drinfel'd (S^{op},H)-module algebras .*

(ii) If A and B are dyslectic Hopf Yetter Drinfel'd (S^{op},H)-modules algebras, then $(A\natural B)^{op} \cong B^{op}\#_S A^{op}$ as dyslectic (S,H)-dimodule algebras .

Proof.

(i) A and B are dyslectic (S,H)-dimodules algebras. Consider the map $\delta : (A\#_S B)^{op} \to B^{op}\natural A^{op}$ by $\delta((a\#b)^o) = b^o\natural a^o$. For all $s \in S$ we have:

$$\begin{aligned}
\delta[((a \leftharpoonup s)\#b)^o] &= b^o\natural(a \leftharpoonup s)^o \\
&= b^o\natural(s_0 \rightharpoonup (S_H(s_1).a))^o \\
&= b^o\natural(s^o \blacktriangleright a^o) \\
&= (b^o \blacktriangleleft s^o)\natural a^o \\
&= (s \rightharpoonup b)^o\natural a^o \\
&= \delta[(a\#(s \rightharpoonup b))^o], \text{ so } \delta \text{ is well-defined.}
\end{aligned}$$

$$\begin{aligned}
\delta(s^o \blacktriangleright (a\#b)^o) &= \delta[(s_0 \rightharpoonup (S_H(s_1)(a\#b)))^o] \\
&= \delta[(s_0 \rightharpoonup [(S_H(s_1).a)\#(S_H(s_2).b)])^o] \\
&= \delta[([s_0 \rightharpoonup (S_H(s_1).a)]\#(S_H(s_2).b))^o] \\
&= (S_H(s_2).b)^o\natural[s_0 \rightharpoonup (S_H(s_1).a)]^o \\
&= (S_H(s_1).b^o)\natural(s_0^o \blacktriangleright a^o) \\
&= ((S_H(s_1).b^o) \blacktriangleleft s_0^o)\natural a^o \\
&= (s^o \blacktriangleright b^o)\natural a^o, \qquad \text{from the relation (18) in [32]} \\
&= s^o \blacktriangleright (b^o\natural a^o) \\
&= s^o \blacktriangleright \delta((a\#b)^o), \text{ so } \delta \text{ is left } S^{op}\text{-linear.}
\end{aligned}$$

$$\begin{aligned}
\delta[h.(a\#b)^o] &= \delta[(h.(a\#b))^o] \\
&= \delta[((h_1.a)\#(h_2.b))^o] \\
&= (h_2.b)^o\natural(h_1.a)^o \\
&= (h_1.b^o)\natural(h_2.a^o) \\
&= h(b^o\natural a^o) \\
&= h\delta[(a\#b)^o],\ \delta \text{ is also } H\text{-linear.}
\end{aligned}$$

Clearly δ is H-linear and H-colinear. From that, δ is a morphism of Hopf Yetter-Drinfel'd (S^{op},H)-modules. We also have:

$$\begin{aligned}
\delta[(a\#b)^o(c\#d)^o] &= \delta[((c\#d)(a\#b))^o] \\
&= \delta[(c(a_1.d)\#a_0b)^o] \\
&= (a_0b)^o\natural(c(a_1.d))^o \\
&= b^oa_0^o\natural(a_1.d^o)c^o \\
&= (b^o\natural a^o)(d^o\natural c^o) \\
&= \delta[(a\#b)^o]\delta[(c\#d)^o],
\end{aligned}$$

δ is an algebra map and it is obvious that it is an isomorphism.

(ii) Let us consider A and B as dyslectic Hopf Yetter Drenfil'd (S^{op},H)-module algebras. Consider the

$$\delta' : (A\natural B)^{op} \to B^{op}\#_S A^{op} \quad by \quad \delta'((a\natural b)^o) = b^o\#a^o$$

for all $a\in A, b\in B$. It is easy to show that δ' is well-defined and it is a homomorphism in $Dys\text{-}_S\mathscr{D}^H$. We'll just show that it is an algebra map. Let $a,c\in A$ and $b,d\in B$:

$$\begin{aligned}
\delta'[(a\natural b)^o(c\natural d)^o] &= \delta'[((c\natural d)(a\natural b))^o] \\
&= \delta'[(ca_0\natural(a_1.d)b)^o] \\
&= ((a_1.d)b)^o\#(ca_0)^o \\
&= (b^o(a_1.d^o))\#a_0^oc^o \\
&= (b^o\#a^o)(d^o\#c^o) \\
&= \delta'((a\natural b)^o)\delta'((c\natural d)^o),
\end{aligned}$$

δ' is an algebra map and it is clearly an isomorphism.

□

Let A be a dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-module algebra. We denote by $\tilde{A}$ its H-opposite algebra which is equal to A as dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-module but with the multiplication given by:

$$\tilde{a}\tilde{b} = \widetilde{b_0(b_1.a)}, \text{ for all } a,b \in A.$$

Note that if A is an algebra in $Dys\text{-}_S\mathscr{D}^H$, then $\overline{A}^{op}$ and $\widetilde{A^{op}}$ are isomorphic as dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-module algebras.
Likewise, if A is an algebra in $Dys\text{-}_{S^{op}}\mathscr{Q}^H$, then $\widetilde{A}^{op}$ and $\overline{A^{op}}$ are isomorphic as algebras in $Dys\text{-}_S\mathscr{D}^H$.

Lemma 2.7.5. *(i) If M is an object of $Dys\text{-}_{S^{op}}\mathscr{Q}^H$ which is faithfully projective as an S^{op}-module, then $End_{S^{op}}(M)^{op} = End_S(M)$ as algebras in $Dys\text{-}_S\mathscr{D}^H$.*

(ii) If M is an object of $Dys\text{-}_S\mathscr{D}^H$ which is faithfully projective as an S-module, then $End_S(M)^{op} = End_{S^{op}}(M)$ as algebras in $Dys\text{-}_{S^{op}}\mathscr{Q}^H$.

Proof.
S^{op} and S have the same elements, similarly for $M \in Dys\text{-}_{S^{op}}\mathscr{Q}^H$, $End_{S^{op}}(M)^{op}$ and $End_S(M)$ have the same elements and we know that

$$f(m \blacktriangleleft s^o) = f(m) \leftharpoonup s$$

In fact we have:

$$f(m) \blacktriangleleft s^o = f(m \blacktriangleleft s^o) \overset{(2.7.4)}{=} f(s \rightharpoonup m) \overset{(2.2.16)}{=} (f \leftharpoonup s)(m).$$

By Femic in [23, Subsection 2.2], since $\mathscr{D}ys\text{-}_S\mathscr{D}^H$ is a braided monoidal category, we deduce an isomorphism of dyslectic (S,H)-dimodules $End_S(M) = {}_SEnd(M)$ in $\mathscr{D}ys\text{-}_S\mathscr{D}^H$, with M finitely generated projective as a left and as a right S-module. Then

$$End_S(M) = {}_SEnd(M) \quad \Longrightarrow \quad (f \leftharpoonup s)(m) \overset{(2.2.17)}{=} f(m) \leftharpoonup s.$$

Therefore

$$f(m) \blacktriangleleft s^o = f(m \blacktriangleleft s^o) \overset{(2.7.4)}{=} f(s \rightharpoonup m) \overset{(2.2.16)}{=} (f \leftharpoonup s)(m) \overset{(2.2.17)}{=} f(m) \leftharpoonup s.$$

$\square$

Proposition 2.7.6. *(i) If A is a dyslectic (S,H)-dimodule Azumaya algebra, then A^{op} is a dyslectic Hopf Yetter-Drindfel'd (S^{op},H)-module Azumaya algebra.*

(ii) If A is a dyslectic Hopf Yetter-Drindfel'd (S^{op},H)-module Azumaya algebra, then A^{op} is a dyslectic (S,H)-dimodule Azumaya algebra.

Proof.

(i) Let A be an Azumaya algebra in $Dys\text{-}_S\mathscr{D}^H$. Then the maps $F : A\#_S\bar{A} \to End_S(A)$ and $G : \bar{A}\#_S A \to \overline{End_S(A)}$ defined in Proposition 2.5.1 are isomorphisms of dyslectic (S,H)-dimodules algebras. Let us consider the maps $F^{op} : A^{op}\natural\widetilde{A^{op}} \to End_{S^{op}}(A^{op})$ and $G^{op} : \widetilde{A^{op}}\natural A^{op} \to \widetilde{End_{S^{op}}(A^{op})}$ defined respectively by

$F^{op}(a^o\natural\widetilde{b^o})(c^o) = a^o c^o_0(c_1.b^o)$ and $G^{op}(\widetilde{a^o}\natural b^o)(c^o) = a^o_0(a_1.c^o)b^o$, for all $a,b,c \in A$.

We have:

$$[F(a\#\bar{b})(c)]^o = (a(b_1.c)b_0)^o = b^o_0(a(b_1.c))^o = b^o_0((b_1.c^o)a^o) = G^{op}(\widetilde{b^o}\natural a^o)(c^o)$$

$$[G(\bar{a}\#b)(c)]^o = ((c_1.a)c_0b)^o = b^o((c_1.a)c_0)^o = b^o(c^o_0(c_1.a)^o) = b^o c^o_0(c_1.a^o) = F^{op}(b^o\natural\widetilde{a^o})(c^o)$$

Since F and G are isomorphisms of dyslectic (S,H)-dimodules algebras, F^{op} and G^{op} are isomorphisms of dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-modules algebras. Hence A^{op} is an Azumaya algebra in $Dys\text{-}_{S^{op}}\mathscr{Q}^H$.

(ii) Let A be an Azumaya algebra in $Dys\text{-}_{S^{op}}\mathscr{Q}^H$ Then the maps

$$V : A\natural\tilde{A} \longrightarrow End_{S^{op}}(A), \quad (a\natural\tilde{b})(c) \longmapsto ac_0(c_1.b)$$

and

$$W : \tilde{A}\natural A \longrightarrow \widetilde{End_{S^{op}}(A)}, \quad (\tilde{a}\natural b)(c) \longmapsto a_0(a_1.c)b,$$

for all $a,b,c \in A$ are isomorphisms of dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-modules algebras see [32] .

Let's consider the maps

$$V^{op} : A^{op}\#\overline{A^{op}} \longrightarrow End_S(A^{op}), \quad (a^o\#\overline{b^o})(c^o) \longmapsto a^o(b_1.c^o)b^o_0$$

and

$$W^{op} : \overline{A^{op}}\#A^{op} \longrightarrow \overline{End_S(A^{op})}, \quad (\overline{a^o}\#b^o)(c^o) \longmapsto (c_1.a^o)c^o_0b^o,$$

for all $a,b,c \in A$. We have:

$$[V(a\natural\tilde{b})(c)]^o = [ac_0(c_1.b)]^o = (c_1.b^o)c^o_0a^o = W^{op}(\overline{b^o}\#a^o)(c^o)$$

$$\text{and} \quad [W(\tilde{a}\natural b)(c)]^o = [a_0(a_1.c)b]^o = b^o(a_1.c^o)a^o_0 = V^{op}(b^o\#\overline{a^o})(c^o)$$

Since V and W are isomorphisms of dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-modules algebras, V^{op} and W^{op} are isomorphisms of dyslectic (S,H)-dimodules algebras, that is, A^{op} is an Azumaya algebra in $Dys\text{-}_S\mathscr{D}^H$.

□

Theorem 2.7.7. *Let H be a Hopf algebra and S be an H-commutative H-dimodule algebra. There is an anti-isomorphism of groups*

$$\chi : BD(S,H) \longrightarrow BQ(S^{op},H) \qquad \textit{given by} \qquad \chi([A]) = \left[\left[A^{op}\right]\right],$$

where $[[A^{op}]]$ represents the class of A^{op} in $BQ(S^{op},H)$.

Proof. Consider two dyslectic (S,H)-dimodules Azumaya algebras A and B which are respectively represented in $BD(S,H)$ by the classes $[A]$ and $[B]$. It is clear that the map χ is well defined (Lemmas 2.7.4 and 2.7.5), and we have:

$$\chi([A].[B]) = \chi([A\#_S B]) = \left[\left[(A\#_S B)^{op}\right]\right] = \left[\left[B^{op}\natural A^{op}\right]\right] = \left[\left[B^{op}\right]\right]\left[\left[A^{op}\right]\right] = \chi[B]\chi[A].$$

So χ is an anti-homomorphism of groups. Clearly, χ is a bijection : its inverse is

$$\chi^{-1} : BQ(S^{op},H) \longrightarrow BD(S,H), \quad [[A]] \longmapsto \left[A^{op}\right],$$

where $[[A]]$ represents the class of A in $BQ(S^{op},H)$. Therefore χ is an anti-isomorphism of groups.

$\square$

Chapter 3

AN ANTI-ISOMORPHISM BETWEEN BRAUER-CLIFFORD-LONG GROUPS $BD(S,H)$ AND $BD(S^{op},H^*)$

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Abstract

For a commutative ring R and a commutative cocommutative Hopf algebra H finitely generated projective as an R-module, Tilborghs in [58], established an anti-isomorphism of groups between the Brauer group $BD(R,H)$ of H-dimodule and the Brauer $BD(R,H^*)$ of H^*-dimodule algebras, where H^* is the linear dual of H. In this paper, we generalize this result by constructing an anti-isomorphism of groups between $BD(S,H)$, the Brauer group of dyslectic (S,H)-dimodule algebras and $BD(S^{op},H^*)$, the Brauer group of dyslectic (S^{op},H^*)-dimodule algebras, where S is an H-commutative H-dimodule algebra and S^{op} is the opposite algebra of S.

Introduction

In [32], Guédénon and Herman introduced a Brauer-Clifford group for the category of dyslectic Hopf Yetter-Drinfel'd (S,H)-modules algebras $BQ(S,H)$, where H is a Hopf algebra with bijective antipode , and S a Hopf Yetter-Drinfel'd module algebra which is H-commutative (or quantum commutative). They used the notion of *dyslectic* in the aim to make braided, the monoidal category $({}_S\mathcal{Q}^H,\tilde{\otimes}_S,\gamma,S)$, where the braiding map

$$\gamma_{M,N}: M\tilde{\otimes}_S N \to N\tilde{\otimes}_S M \quad \text{is given by} \quad \gamma_{M,N}(m\tilde{\otimes}_S n) = n_0\tilde{\otimes}_S(n_1 m).$$

When H is commutative and cocommutative, a Hopf Yetter-Drinfel'd H-module becomes an H-dimodule, from this we study in [33] the Brauer-Clifford group $BD(S,H)$ of dyslectic (S,H)-

dimodules algebras , where the braiding map making braided the monoidal category $({}_S\mathscr{D}^H, \tilde{\otimes}_S, \gamma', S)$,

$$\gamma'_{M,N} : M\tilde{\otimes}_S N \to N\tilde{\otimes}_S M \quad \text{is given by} \quad \gamma'_{M,N}(m\tilde{\otimes}_S n) = (m_1 n)\tilde{\otimes}_S m_0,$$

and where the H-commutativity of S comes from γ'. We have concluded our study by establishing an anti-isomorphism of groups between our Brauer group $BD(S,H)$ of dyslectic (S,H)-dimodules algebras and the Brauer group $BQ(S^{op},H)$ of dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-modules algebras, where S^{op} is the natural opposite algebra of S.

The Brauer group $BD(S,H)$ of dyslectic (S,H)-dimodule algebras is a generalization of the Brauer group $BD(R,H)$, constructed by Long in [40], of dimodule algebras for a commutative ring R and a commutative and cocommutative, finitely generated projective Hopf algebra H over R. If H is finitely generated projective, H^* is also a Hopf algebra and $H \cong H^{**}$ as Hopf algebras, and, Tilborghs established in [58], an anti-isomorphism of groups between the Brauer-Long groups $BD(R,H)$ and $BD(R,H^*)$. Our aim in this paper is to generalize this result to the Brauer-Clifford-Long group $BD(S,H)$ of the category of dyslectic (S,H)-dimodule algebras and the Brauer-Clifford-Long group $BD(S^{op},H^*)$ of the category of dyslectic (S^{op},H^*)-dimodule algebras, where S is H-commutative as an H-dimodule algebra.

After recalling the basic notions and definitions of Hopf algebra in the first part, we show in the second part that if S is H-commutative as an H-dimodule algebra, then its natural opposite S^{op} is H^*-commutative as an H^*-dimodule algebra. Accordinding to [33] the category $Dys\text{-}{}_{S^{op}}\mathscr{D}^{H^*}$ of dyslectic (S^{op},H^*)-dimodule algebras is a braided monoidal category and we can condider its Brauer group $BD(S^{op},H^*)$. In the third part, we end our paper by establishing an anti-isomorphism of group between $BD(S,H)$ and $BD(S^{op},H^*)$.

For more details on Hopf algebras and Brauer groups, we refer to the literature, see for example [1], [14], [44], [56].

3.1 Preliminaries and notations

Let H be a Hopf algebra over a commutative ring R. We denote its comultiplication by $\Delta : H \to H\otimes H$, its antipode by $S_H : H \to H$ and its counit by $\varepsilon : H \to R$. We will use Sweedler-Heyneman notation, omitting sums, so we write $\Delta(h) = h_1 \otimes h_2$.
For a Hopf algebra with comultiplication Δ, Δ^{cop} is defined by

$$\Delta^{cop}(h) = h_2 \otimes h_1.$$

A Hopf algebra H is said to be *cocommutative* if

$$h_1 \otimes h_2 = h_2 \otimes h_1,$$

for all $h \in H$. We will require a sequence of definitions, all of which are standard. An R-algebra A is an *H-module algebra* if A is a left H-module such that

$$h.(ab) = (h_1.a)(h_2.b) \text{ and } h.1_A = \varepsilon(h)1_A, \text{ for all } a,b \in A, h \in H. \tag{3.1.1}$$

H acts trivially on A when $h.a=\varepsilon(h)a$ for all $h\in H$ and $a\in A$. A *homomorphism of H-module algebras* is a homomorphism of H-modules which is also a homomorphism of R-algebras. If A is an H-module algebra, then the smash product algebra $A\#H$ is the R-module $A\otimes H$ with multiplication

$$(a\otimes h)(a'\otimes h')=[a(h_1.a')]\otimes(h_2h'),\text{ for all } a,a'\in A \text{ and } h,h'\in H. \tag{3.1.2}$$

An R-module M is a left $A\#H$-module if it is a left A-module and a left H-module for which

$$h(am)=(h_1.a)(h_2m),\text{ for all } h\in H, a\in A \text{ and } m\in M. \tag{3.1.3}$$

We will write ${}_{A\#H}\mathcal{M}$ for the category of left $A\#H$-modules. It was observed in [31, Theorem 2.2] that if H is cocommutative and A is a commutative H-module algebra, then $({}_{A\#H}\mathcal{M},\otimes_A,A)$ is a symmetric monoidal category.
If H is a Hopf algebra over R. An R-module M is a *right H-comodule* if there exists an R-linear map $\rho_M: M\to M\otimes H$ satisfying the relations

$$(\rho_M\otimes id_H)\circ\rho_M=(id_M\otimes\Delta)\circ\rho_M \quad\text{ and }\quad (id_M\otimes\varepsilon)\circ\rho_M=id_M.$$

In Sweedler notation, we write

$$\rho_M(m)=m_0\otimes m_1 \text{ for all } m\in M,$$

and the right H-comodule conditions on M are

$$m_{00}\otimes m_{01}\otimes m_1=m_0\otimes m_{11}\otimes m_{12}=m_0\otimes m_1\otimes m_2 \tag{3.1.4}$$
$$\text{and } m_0\varepsilon(m_1)=m,\text{ for all } m\in M.$$

H coacts trivially on M when $m_0\otimes m_1=m_0\otimes 1_H$, for all $m\in M$. Let M and N be right H-comodules. A *homomorphism of right H-comodules* (aka. a right H-colinear map) is an R-linear map $f: M\to N$ such that $\rho_N\circ f=(f\otimes id_H)\circ\rho_M$. In Sweedler notation, this is equivalent to

$$f(m)_0\otimes f(m)_1=f(m_0)\otimes m_1,\text{ for all } m\in M. \tag{3.1.5}$$

If M and N are right H-comodules, then $M\otimes N$ is a right H-comodule under the codiagonal coaction:

$$(m\otimes n)_0\otimes(m\otimes n)_1=(m_0\otimes n_0)\otimes(m_1n_1),\quad m\in M, n\in N. \tag{3.1.6}$$

An R-algebra A is an *H-comodule algebra* if A is a right H-comodule and the multiplication in A satisfies

$$(ab)_0\otimes(ab)_1=(a_0b_0)\otimes(a_1b_1) \text{ and } \rho_A(1_A)=1_A\otimes 1_H,\text{ for all } a,b\in A. \tag{3.1.7}$$

A *homomorphism of H-comodule algebras* is a homomorphism of H-comodules which is also a homomorphism of R-algebras.
Let A be a right H-comodule algebra. An R-module M is an *(A,H)-Hopf module* if M is both a left A-module and a right H-comodule, with the property

$$(am)_0\otimes(am)_1=(a_0m_0)\otimes(a_1m_1),\text{ for all } a\in A, m\in M. \tag{3.1.8}$$

A *homomorphism of* (A,H)*-Hopf modules* is a left A-linear map which is also a right H-colinear map. We will write ${}_A\mathscr{M}^H$ for the category of (A,H)-Hopf modules. This category is dual to ${}_{A\#H}\mathscr{M}$. When H is commutative and A is a commutative H-comodule algebra, $({}_A\mathscr{M}^H, \otimes_A, A)$ is a symmetric monoidal category [31].

For a left H-module M, we denote by $\lambda_M : H\otimes M \to M$ the left H-action on M. An H-*dimodule* is an R-module M which is also a left H-module and a right H-comodule such that $\rho_M \circ \lambda_M = (\lambda_M \otimes id_H)\circ(id_H \otimes \rho_M)$, that is:

$$(hm)_0 \otimes (hm)_1 = (hm_0)\otimes m_1, \text{ for all } h \in H, m \in M. \tag{3.1.9}$$

If M and N are H-dimodules, an R-linear map $f : M \to N$ is said to be an *H-dimodule homomorphism* if it is simultaneously an H-module homomorphism and an H-comodule homomorphism.

An *H-dimodule algebra* is an R-algebra which is an H-dimodule so that it is an H-module algebra and an H-comodule algebra satisfying the relation (3.1.9). An *H-dimodule algebra homomorphism* between two H-dimodule algebras A and B is an R-linear map $A \to B$ which is simultaneously an H-dimodule homomorphism and an R-algebra homomorphism.

We denote the category of H-dimodules by $\mathscr{D}^H$. For H-dimodules M and N, the tensor product $M\otimes N$ has an H-module structure given by

$$h(m\otimes n) = (h_1 m)\otimes(h_2 n), \text{for all } m \in M, n \in N, \tag{3.1.10}$$

and an H-comodule structure given by

$$(m\otimes n)_0 \otimes (m\otimes n)_1 = (m_0 \otimes n_0)\otimes(m_1 n_1), \text{ for all } m \in M, n \in N. \tag{3.1.11}$$

These H-structures satisfy the compatibility condition (3.1.9) and make $M\otimes N$ an H-dimodule, denoted by $M\tilde{\otimes}N$.

Let H be commutative and cocommutative. For H-dimodules M and N, there exists an H-dimodule isomorphism $\gamma_{M,N}$ from $M\tilde{\otimes}N$ to $N\tilde{\otimes}M$ defined by (see [59])

$$\gamma_{M,N}(m\tilde{\otimes}n) = (m_1 n)\tilde{\otimes}m_0, \text{ for all } m \in M, n \in N, \tag{3.1.12}$$

with inverse is

$$\gamma^{-1}_{M,N}(n\tilde{\otimes}m) = m_0\tilde{\otimes}(S_H(m_1)n), \text{ for all } m \in M, n \in N. \tag{3.1.13}$$

A monoidal category $(\mathscr{C},\otimes)$ is braided if there are natural isomorphisms $\gamma_{M,N} : M\otimes N \cong N\otimes M$ in $\mathscr{C}$ for all $M,N \in \mathscr{C}$, such that the hexagonal coherence conditions given in Definition 1.6.13 are satisfied, that is:

$$\gamma_{M\otimes N,P} = (\gamma_{M,P}\otimes 1)\circ(1\otimes\gamma_{N,P})$$

and

$$\gamma_{M,N\otimes P} = (1\otimes\gamma_{M,P})\circ(\gamma_{M,N}\otimes 1),$$

for all $M,N,P \in \mathscr{C}$.

The natural opposite algebra A^{op} of A is an isomorphic copy of A as an R-module with the multiplication given by

$$a^o b^o = (ba)^o \qquad \text{for all } a,b \in A. \tag{3.1.14}$$

Let A be an H-dimodule algebra, the H-opposite $\bar{A}$ of A is an isomorphic copy of A as an H-dimodule and the multiplication on $\bar{A}$ is defined by

$$\bar{a}\bar{b} = \overline{(a_1.b)a_0}, \qquad \text{for all } a,b \in A. \tag{3.1.15}$$

Let A be an H-dimodule algebra. The algebras A^{op} and $\bar{A}$ are H-dimodules algebras (see [40]).

According to [59, EXAMPLE 3.11], $(\mathscr{D}^H, \tilde{\otimes}_R, \gamma_{M,N}, R)$ is a braided monoidal category. Note that an H-dimodule algebra is just an algebra in the braided monoidal category $\mathscr{D}^H$.

From Long in [40], we can form the *Brauer group of* $\mathscr{D}^H$ denoted by $BD(R,H)$.

From now on, H is commutative cocommutative and finitely generated projective as an R-module. Let $\{h_i, h_i^*;\ i \in I\}$ be a finite dual basis of H as an R-module (where I is a finite set of index), that is,

$$\forall h \in H, \quad h = \sum_i \langle h_i^*, h\rangle h_i \quad \text{ and } \quad \forall h^* \in H^*, \quad h^* = \sum_i \langle h^*, h_i\rangle h_i^*$$

Refering to Long in [40] and Pareigis in [47], the dual H^* of H is also a commutative and cocommutative Hopf algebra finitely generated projective as an R-module: its comultiplication

$$\Delta^* : H^* \to (H \otimes H)^* \cong H^* \otimes H^* \quad \text{ is given by}$$

$$[\Delta^*(h^*)](h \otimes h') = (h^*_{(1)} \otimes h^*_{(2)})(h \otimes h') = h^*_{(1)}(h)h^*_{(2)}(h') = h^*(hh');$$

$$\text{its counit} \qquad \varepsilon^* : H^* \to R, \quad \text{is given by} \quad \varepsilon^*(h^*) = h^*(1_H),$$

$$\text{and its antipode} \qquad S_{H^*} : H^* \to H^*, \quad \text{is given by} \quad S_{H^*}(h^*) = h^* \circ S_H, \quad for \ \ h^* \in H^*$$

Note that $H^{**} \cong H$ as Hopf algebras. Every H-dimodule M is an H^*-dimodule: the left H^*-action $\alpha_M^* : H^* \otimes M \to M$ is given by:

$$\alpha_M^*(h^* \otimes m) = h^*.m = m_0 h^*(m_1), \quad \text{for all } h^* \in H^*, m \in M, \tag{3.1.16}$$

and the right H^*-comodule structure $\rho_M^* : M \to M \otimes H^*$, is given by

$$\rho_M^*(m) = m_{(0)} \otimes m^*_{(1)} = \sum_i h_i m \otimes h_i^*, \tag{3.1.17}$$

where (see [58, Proposition: 3,(2)]),

$$m_{(0)} m^*_{(1)}(h) = hm \quad \text{for all } h \in H, m \in M \tag{3.1.18}$$

Since R is a commutative ring, $m_{(0)}m^*_{(1)}(h) = m^*_{(1)}(h)m_{(0)} = hm$.

The compatibility condition for M to be an H^*-dimodule is given by

$$(h^*.m)_{(0)} \otimes (h^*.m)^*_{(1)} = (h^*.m_{(0)}) \otimes m^*_{(1)} \quad \text{for all } h \in H, m \in M. \tag{3.1.19}$$

Since H^* is commutative and cocommutative, we can consider the category $\mathscr{D}^{H^*}$ of H^*-dimodules: the objects are the H^*-dimodules and the morphisms are the H^*-dimodule homomorphisms (the R-linear maps that are simultaneously left H^*-linear and right H^*-colinear). We also have the Brauer group $BD(R,H^*)$ of H^*-dimodule algebras. We recall that the braiding map defining the Brauer group $BD(R,H^*)$, $\gamma^*_{M,N} : M\tilde{\otimes}N \to N\tilde{\otimes}M$ (for M and N two H^*-dimodules) is given by

$$\gamma^*_{M,N}(m\tilde{\otimes}n) = (m^*_{(1)}.n)\tilde{\otimes}m_{(0)}, \tag{3.1.20}$$

with inverse $\gamma^{*^{-1}}_{M,N} : N\tilde{\otimes}M \to M\tilde{\otimes}N$ defined by

$$\gamma^{*^{-1}}_{M,N}(n\tilde{\otimes}m) = m_{(0)}\tilde{\otimes}(S_{H^*}(m^*_{(1)}).n), \quad \text{for all } m \in M, n \in N. \tag{3.1.21}$$

With the H^*-dimodule structures defined above, according to Tilborghs in [58], if A is an H-dimodule algebra, then it is also an H^*-dimodule algebra and so is its opposite algebra A^{op}. Also if A is an H-Azumaya H-dimodule algebra, its opposite A^{op} is an H^*-Azumaya H^*-dimodule algebra.

By Tilborghs in [58], there is an anti-isomorphism of group between $BD(R,H)$ and $BD(R,H^*)$ mapping $[A]$ to $\overline{[A^{op}]}$ where $[A]$ represents the equivalence class of an H-Azumaya H-dimodule algebra A. Our aim in this paper is to extend this result to the Brauer group of dyslectic (S,H)-dimodules.

3.2 The category of (S^{op},H^*)-dimodules

Let S be an H-dimodule algebra. An (S,H)-dimodule M is a left S-module (we denote by $\rightharpoonup$ the left S-action) and an H-dimodule satisfying the compatibility conditions (3.1.3) and (3.1.8).

Equivalently, M is a left $S\#H$-module and a right Hopf (S,H)-module for which the relation (2.1.9) is satisfied. An (R,H)-dimodule is just an H-dimodule. Furthermore, note that if S is an H-dimodule algebra, then S is an (S,H)-dimodule : the left S-action is given by $s \rightharpoonup s' = ss'$, for all $s,s' \in S$.

An (S,H)-*dimodule homomorphism* is an H-dimodule map which is also left S-linear. We denote by ${}_S\mathscr{D}^H$, the category consisting of (S,H)dimodules and (S,H)-dimodules homomorphisms.

Let S be an H-module algebra. We say that S is H-*commutative* if

$$ss' = (s_1.s')s_0, \quad \text{for all } s,s' \in S. \tag{3.2.1}$$

If S is an H-commutative H-dimodule algebra, then for every left S-action on $M \in {}_S\mathscr{D}^H$, there is a corresponding right S-action denoted by $\leftharpoonup$ and defined by

$$m \leftharpoonup s = (m_1.s) \rightharpoonup m_0, \text{ for all } s \in S, m \in M. \tag{3.2.2}$$

With this S-action, M can be seen as an S-S-bimodule. The right and the left S-action are related by

$$s \rightharpoonup m = m_0 \leftharpoonup (S_H(m_1).s), \text{ for all } s \in S, m \in M. \tag{3.2.3}$$

Note also that we have

$$h(m \leftharpoonup s) = (h_1 m) \leftharpoonup (h_2.s) \tag{3.2.4}$$

and

$$(m \leftharpoonup s)_0 \otimes (m \leftharpoonup s)_1 = (m_0 \leftharpoonup s_0) \otimes (m_1 s_1), \text{ for all } h \in H, m \in M, s \in S. \tag{3.2.5}$$

Let S be an H-commutative H-dimodule algebra. Then for M and N in ${}_S\mathscr{D}^H$, we can endow the tensor product $M \otimes_S N$ with the following S-action, H-action and H-coaction:

$$s \rightharpoonup (m \tilde{\otimes}_S n) = (s \rightharpoonup m) \tilde{\otimes}_S n, \tag{3.2.6}$$

$$h(m \tilde{\otimes}_S n) = (h_1 m) \tilde{\otimes}_S (h_2 n) \tag{3.2.7}$$

$$\text{and} \quad (m \tilde{\otimes}_S n)_0 \otimes (m \tilde{\otimes}_S n)_1 = (m_0 \tilde{\otimes}_S n_0) \otimes (m_1 n_1), \tag{3.2.8}$$

for all $h \in H, s \in S, m \in M,$ and $n \in N$. Note that we have

$$(m \tilde{\otimes}_S n) \leftharpoonup s = m \tilde{\otimes}_S (n \leftharpoonup s), \text{ for all } m \in M, n \in N, s \in S. \tag{3.2.9}$$

Throughout this section, H is a commutative and cocommutative Hopf algebra and S is an H-commutative H-dimodule algebra.

Lemma 3.2.1. *Let S be an H-commutative H-dimodule algebra. The opposite algebra S^{op} of S is an H^*-commutative H^*-dimodule algebra.*

Proof. For all $s, t \in S$, we have,

$$s^o t^o = (ts)^o = [(t_1.s)t_0]^o = t_0^o(t_1.s^o) = t_0^o[s^*_{(1)}(t_1).s^o_{(0)}] = [t_0^o s^*_{(1)}(t_1)]s^o_{(0)} = (s^*_{(1)}.t^o)s^o_{(0)}$$

$s^o t^o = (s^*_{(1)}.t^o)s^o_{(0)}$, then S^{op} is an H^*-commutative H^*-dimodule algebra.

□

Since S^{op} is an H^*-commutative H^*-dimodule algebra, we can form the category ${}_{S^{op}}\mathscr{D}^{H^*}$ of (S^{op}, H^*)-dimodules: its morphisms are the left S^{op}-linear, H^*-linear and H^*-colinear maps.

Let M be a left (S,H)-dimodule. Then M becomes a left S^{op}-module when we set :

$$s^o \triangleright m = m \leftharpoonup s \quad \text{for all } m \in M, s \in S. \tag{3.2.10}$$

Lemma 3.2.2. *M is an (S,H)-dimodule if and only if it is an (S^{op},H^*)-dimodule.*

Proof. Consider M an (S,H)-dimodule. By (3.2.10), M is a left S^{op}-module. Let $h^*\in H^*, m\in M$ and $s\in S$,

$$\begin{aligned}
h^*.(s^o \triangleright m) &= h^*.(m \leftharpoonup s)\\
&= (m \leftharpoonup s)_0 h^*((m \leftharpoonup s)_1)\\
&= (m_0 \leftharpoonup s_0)h^*(m_1 s_1)\\
&= (m_0 \leftharpoonup s_0)h^*_{(1)}(m_1)h^*_{(2)}(s_1)\\
&= (m_0 h^*_{(1)}(m_1)) \leftharpoonup (s_0 h^*_{(2)}(s_1))\\
&= (h^*_{(1)}.m) \leftharpoonup (h^*_{(2)}.s)\\
&= (h^*_{(1)}.s)^o \triangleright (h^*_{(2)}m)\\
&= (h^*_{(1)}.s^o) \triangleright (h^*_{(2)}m).
\end{aligned}$$

Let $m\in M, s\in S$; we have:

$$(s^o \triangleright m)_{(0)} \otimes (s^o \triangleright m)^*_{(1)} = (m \leftharpoonup s)_{(0)} \otimes (m \leftharpoonup s)^*_{(1)} = (m_0 \leftharpoonup s_0) \otimes (m_1 s_1)^* = (s^o_{(0)} \triangleright m_{(0)}) \otimes (s^*_{(1)} m^*_{(1)}).$$

Then the H^*-coaction and the S^{op}-action on M are compatible. We know that M is an H^*-dimodule.

Now, let M be an (S^{op},H^*)-dimodule. Clearly, M is a left S-module and an H-dimodule. For all $m\in M, s\in S$ and $h\in H$, we have

$$\begin{aligned}
h(s \rightharpoonup m) &= h[m_0 \leftharpoonup (S_H(m_1).s)]\\
&= h[(S_H(m_1).s)^o \triangleright m_0]\\
&= [h_1.(S_H(m_1).s)]^o \triangleright (h_2 m_0)\\
&= [S_H(m_1).(h_1.s)]^o \triangleright (h_2 m_0)\\
&= (h_2 m_0) \leftharpoonup [S_H(m_1).(h_1.s)]\\
&= (h_2 m)_0 \leftharpoonup [S_H((h_2 m)_1).(h_1.s)]\\
&= (h_1.s) \rightharpoonup (h_2 m)
\end{aligned}$$

$$\begin{aligned}
(s \rightharpoonup m)_0 \otimes (s \rightharpoonup m)_1 &= [m_0 \leftharpoonup S_H(m_1).s]_0 \otimes [m_0 \leftharpoonup S_H(m_1).s]_1\\
&= [(S_H(m_1).s)^o \triangleright m_0]_0 \otimes [(S_H(m_1).s)^o \triangleright m_0]_1\\
&= [(S_H(m_1).s^o_0) \triangleright m_{00}] \otimes (s_1 m_{01})\\
&= [m_{00} \leftharpoonup (S_H(m_{01}).s_0)] \otimes (s_1 m_1)\\
&= (s_0 \rightharpoonup m_0) \otimes (s_1 m_1)
\end{aligned}$$

Then M is an (S,H)-dimodule.

□

Let $M \in {}_{S^{op}}\mathscr{D}^{H^*}$. Since S^{op} is an H^*-commutative H^*-dimodule algebra, by (3.2.2), there is a corresponding right S^{op}-action on M defined by:

$$m \triangleleft s^o = (m^*_{(1)}.s^o) \triangleright m_{(0)}, \quad \text{for all } m \in M, s \in S. \tag{3.2.11}$$

This allows us to view M as an S^{op}-S^{op}-bimodule. Note that the right S^{op}-action and the right S-action on M are related by:

$$m \triangleleft s^o = (s_1 m) \leftharpoonup s_0, \quad \text{for all } m \in M, s \in S. \tag{3.2.12}$$

since we have:

$$m \triangleleft s^o = (m^*_{(1)}.s^o) \triangleright m_{(0)} = (m^*_{(1)}(s_1).s^o_0) \triangleright m_{(0)} = s^o_0 \triangleright (m^*_{(1)}(s_1)m_{(0)}) = s^o_0 \triangleright (s_1 m) = (s_1 m) \leftharpoonup s_0.$$

Note also that by (3.2.3), the left S^{op}-action and the right S^{op}-action on M are related by

$$s^o \triangleright m = m_{(0)} \triangleleft (S^*_H(m^*_{(1)}).s^o), \quad \text{for all } m \in M, s \in S. \tag{3.2.13}$$

By (3.2.4) and (3.2.5) we have, for all $m \in M, s \in S$ and $h^* \in H^*$,

$$h^*(m \triangleleft s^o) = (h^*_{(1)}.m) \triangleleft (h^*_{(2)}.s^o) \tag{3.2.14}$$

$$\text{and } (m \triangleleft s^o)_0 \otimes (m \triangleleft s^o)^*_{(1)} = (m_{(0)} \triangleleft s^o_{(0)}) \otimes (m^*_{(1)} s^*_{(1)}) \tag{3.2.15}$$

For all M and N in ${}_{S^{op}}\mathscr{D}^{H^*}$, we can endow the tensor product $M \tilde{\otimes}_{S^{op}} N$ with the following S^{op}-action, H^*-module and H^*-comodule structures:

$$s^o \triangleright (m \tilde{\otimes}_{S^{op}} n) = (s^o \triangleright m) \tilde{\otimes}_{S^{op}} n, \tag{3.2.16}$$

$$h^*(m \tilde{\otimes}_{S^{op}} n) = (h^*_{(1)} m) \tilde{\otimes}_{S^{op}} (h^*_{(2)} n), \tag{3.2.17}$$

$$\text{and} \quad (m \tilde{\otimes}_{S^{op}} n)_{(0)} \otimes (m \tilde{\otimes}_{S^{op}} n)^*_{(1)} = (m_{(0)} \tilde{\otimes}_{S^{op}} n_{(0)}) \otimes (m^*_{(1)} n^*_{(1)}); \tag{3.2.18}$$

for all $m \in M, n \in N, s \in S$ and $h^* \in H^*$. We note that

$$(m \tilde{\otimes}_{S^{op}} n) \triangleleft s^o = m \tilde{\otimes}_{S^{op}} (n \triangleleft s^o). \tag{3.2.19}$$

With these structures defined above, $M \tilde{\otimes}_{S^{op}} N$ is an (S^{op}, H^*)-dimodule.

According to [33], $({}_{S^{op}}\mathscr{D}^{H^*}, \tilde{\otimes}_{S^{op}}, S^{op})$ is a monoidal category.

Let M be an (S,H)-dimodule. M is said to be a *dyslectic* (S,H)*-dimodule* if

$$h_M \circ \gamma_{M,S} \circ \gamma_{S,M} = h_M,$$

where $h_M : S \otimes M \to M$ denotes the left S-action on M. In other words, an object M of ${}_S\mathscr{D}^H$ is dyslectic if and only if (see [33, (31) and (32)])

$$s \rightharpoonup m = [(s_1m)_1.s_0] \rightharpoonup (s_1m)_0 = (m_1.s_0) \rightharpoonup (s_1m_0), \tag{3.2.20}$$

for $s \in S, m \in M$. According to [33], its equivalent right condition is

$$m \leftharpoonup s = s_0 \rightharpoonup (S_H(s_1)m). \tag{3.2.21}$$

A *dyslectic* (S,H)*-dimodule homomorphism* is an (S,H)-dimodule homomorphism between dyslectic (S,H)-dimodules.

From Lemma 4.1 and Lemma 4.2 of [33], the fact that the conditions (3.2.20) and (3.2.21) are satisfied induces a well-defined braiding map

$$\gamma_{M,N} : M\tilde{\otimes}_S N \to N\tilde{\otimes}_S M; \quad m\tilde{\otimes}_S n \mapsto (m_1 n)\tilde{\otimes}_S m_0$$

with inverse

$$\gamma^{-1}_{M,N} : N\tilde{\otimes}_S M \to M\tilde{\otimes}_S N; \quad n\tilde{\otimes}_S m \mapsto m_0\tilde{\otimes}_S (S_H(m_1)n),$$

which is also well-defined.

$Dys\text{-}_S\mathscr{D}^H$ denotes the category of dyslectic (S,H)-dimodules with (S,H)-dimodules homomorphisms, it is a full subcategory of $_S\mathscr{D}^H$. For $M,N \in Dys\text{-}_S\mathscr{D}^H$, by [33, Lemma 4.3], $M\tilde{\otimes}_S N$ is also in $Dys\text{-}_S\mathscr{D}^H$, and by [33, Theorem 4.4], $(Dys\text{-}_S\mathscr{D}^H, \tilde{\otimes}_S, S, \gamma_{M,N})$ is a braided monoidal category.

Since S^{op} is H^*-commutative, we can consider the category of dyslectic (S^{op},H^*)-dimodule with (S^{op},H^*)-dimodule homomorphisms, it is a full subcategory of $_{S^{op}}\mathscr{D}^{H^*}$. It follows from [33, Theorem 4.4] that $(Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}, \tilde{\otimes}_{S^{op}}, S^{op}, \gamma^*_{M,N})$ is a braided monoidal category. We recall that $\gamma^*_{M,N}$ is defined from $M\tilde{\otimes}_{S^{op}}N$ to $N\tilde{\otimes}_{S^{op}}M$ by

$$\gamma^*_{M,N}(m\tilde{\otimes}_{S^{op}} n) = (m^*_{(1)}.n)\tilde{\otimes}_{S^{op}} m_{(0)}$$

for all objects M and N in $Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$.

Lemma 3.2.3. *i) M is a dyslectic (S,H)-dimodule if and only if M is a dyslectic (S^{op},H^*)-dimodule.*

ii) The two braiding maps are related by $\gamma_{M,N} = \tau_{M,N} \circ \gamma^*_{N,M} \circ \tau_{M,N}$, *where $M,N \in Dys\text{-}_S\mathscr{D}^H$ and $\tau_{M,N}$ is the flip map.*

Proof.

$i)$ Let M be an object of $Dys\text{-}_S\mathscr{D}^H$. For all $m \in M, s \in S$, we have

$$\begin{aligned}
s^o \rhd m = m \leftharpoondown s &\overset{(3.2.2)}{=} (m_1.s) \rightharpoonup m_0 \\
&\overset{(3.2.20)}{=} [m_1.(m_2.s_0)] \rightharpoonup (s_1 m_0) \\
&= [m_2.(m_1.s_0)] \rightharpoonup (s_1 m_0) \\
&\overset{(3.1.18)}{=} [m_2.(s_{(00)} s^*_{(01)}(m_1))] \rightharpoonup [m_{(00)} m^*_{(01)}(s_1)] \\
&= [m_2.(s_{(00)} m^*_{(01)}(s_1))] \rightharpoonup [m_{(00)} s^*_{(01)}(m_1)] \\
&= [m_2.(s_{(00)} m^*_{(1)}(s_{01}))] \rightharpoonup [m_{(00)} s^*_{(1)}(m_{01})] \\
&\overset{(3.1.16)}{=} [m_2.(m^*_{(1)}.s_0)] \rightharpoonup (s^*_{(1)} m_0) \\
&= [m_1.(m^*_{(2)}.s_0)] \rightharpoonup (s^*_{(1)} m_0) \\
&= [m_{01}.(m^*_{(1)}.s_0)] \rightharpoonup (s^*_{(1)} m_{00}) \\
&\overset{(2.1.9)}{=} [(s^*_{(1)} m_0)_1.(m^*_{(1)}.s_0)] \rightharpoonup (s^*_{(1)} m_0)_0 \\
&\overset{(3.2.2)}{=} (s^*_{(1)} m_0) \leftharpoondown (m^*_{(1)}.s_0) \\
&= (m^*_{(1)}.s^o_0) \rhd (s^*_{(1)} m_0)
\end{aligned}$$

then $M \in Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$. Now let M be an object of $Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$. For every $m \in M, s \in S$, we have:

$$\begin{aligned}
m \leftharpoondown s = s^o \rhd m &= m_{(0)} \lhd (S^*_H(m^*_{(1)}).s^o) \\
&= (S^*_H(m^*_{(1)}).s^o)_{(0)} \rhd [S^*_H(S^*_H(m^*_{(1)}).s^o)^*_{(1)} m_{(0)}] \\
&= (S^*_H(m^*_{(1)}).s^o_{(0)}) \rhd (S^*_H(s^*_{(1)}) m_{(0)}) \\
&= [s^o_{(00)} S^*_H(m^*_{(1)})(s_{(01)})] \rhd [m_{(00)} S^*_H(s^*_{(1)})(m_{(01)})] \\
&= [s^o_{(00)} m^*_{(1)}(S_H(s_{(01)}))] \rhd [m_{(00)} s^*_{(1)}(S_H(m_{(01)}))] \\
&= [s^o_{(00)} s^*_{(1)}(S_H(m_{(01)}))] \rhd [m_{(00)} m^*_{(1)}(S_H(s_{(01)}))] \\
&= [S_H(m_1).s^o_0] \rhd [S_H(s_1) m_0] \\
&= [S_H(m_1).s_0]^o \rhd [S_H(s_1) m_0] \\
&= [S_H(s_1) m_0] \leftharpoondown [S_H(m_1).s_0] \\
&= (S_H(s_1) m)_0 \leftharpoondown [S_H((S_H(s_1) m)_1).s_0] \\
&= s_0 \rightharpoonup (S_H(s_1) m)
\end{aligned}$$

We deduce that, $(M \in Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}) \Rightarrow (M \in Dys\text{-}_S\mathscr{D}^H)$.

$ii)$ Let $m \in M, n \in N$, then

$$\begin{aligned}
\gamma_{M,N}(m \tilde{\otimes}_S n) &= (m_1.n) \tilde{\otimes}_S m_0 \\
&= (n_{(0)} n^*_{(1)}(m_1)) \tilde{\otimes}_{S^{op}} m_0 \\
&= n_{(0)} \tilde{\otimes}_{S^{op}} (n^*_{(1)}(m_1) m_0) \\
&= n_{(0)} \tilde{\otimes}_{S^{op}} (n^*_{(1)} m) \\
&= \tau_{M,N}((n^*_{(1)} m) \tilde{\otimes}_{S^{op}} n_{(0)}) \\
&= (\tau_{M,N} \circ \gamma^*_{N,M})(n \tilde{\otimes}_{S^{op}} m) \\
&= (\tau_{M,N} \circ \gamma^*_{N,M} \circ \tau_{M,N})(m \tilde{\otimes}_{S^{op}} n)
\end{aligned}$$

□

Remark 3.2.4. *Note that if M is a dyslectic (S,H)-dimodule, then*

$$m \triangleleft s^o = s \rightharpoonup m, \quad \text{for all } m \in M, s \in S. \tag{3.2.22}$$

In fact,

$$m \triangleleft s^o \overset{(3.2.12)}{=} (s_1 m) \leftharpoonup s_0 \overset{(3.2.2)}{=} ((s_1 m)_1.s_0) \rightharpoonup (s_1 m)_0 \overset{(2.1.9)}{=} (m_1.s_0) \rightharpoonup (s_1 m_0) \overset{(3.2.20)}{=} s \rightharpoonup m. \quad \square$$

Proposition 3.2.5. *We have an isomorphism of categories $\mathscr{F} : Dys\text{-}_S\mathscr{D}^H \to Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$.*

Proof. For all $M \in Dys\text{-}_S\mathscr{D}^H$, we have $\mathscr{F}(M) = M \in Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$ from Lemma 3.2.2 and 3.2.3. Let $M, N \in Dys\text{-}_S\mathscr{D}^H$ and $f : M \to N$, a morphism of $Dys\text{-}_S\mathscr{D}^H$. We also have $\mathscr{F}(f) = f \in Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$; since, for all $s \in S, m \in M$;

$$\begin{aligned} f(s^o \triangleright m) &= f(m \leftharpoonup s) \\ &= f((m_1.s) \rightharpoonup m_0) \\ &= (m_1.s) \rightharpoonup f(m_0) \\ &= (f(m)_1.s) \rightharpoonup f(m)_0 \\ &= f(m) \leftharpoonup s \\ &= s^o \triangleright f(m); \end{aligned}$$

that is f is S^{op}-linear.

$$f(h^* m) = f(m_0 h^*(m_1)) = f(m_0) h^*(m_1) = f(m)_0 h^*(f(m)_1) = h^* f(m)$$

and

$$\begin{aligned} f(m)_{(0)} \otimes f(m)^*_{(1)} &= \textstyle\sum_i h_i.f(m) \otimes h_i^* \\ &= \textstyle\sum_i f(h_i m) \otimes h_i^* \\ &= f(m_{(0)}) \otimes m^*_{(1)}; \end{aligned}$$

that is, f is left H^*-linear and right H^*-colinear.

The insverse functor of $\mathscr{F}$ is the functor $\mathscr{G} : Dys\text{-}_{S^{op}}\mathscr{D}^{H^*} \to Dys\text{-}_S\mathscr{D}^H$ such that, for all $M \in Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$; $\mathscr{G}(M) = M \in Dys\text{-}_S\mathscr{D}^H$.

□

3.3 Anti-isomorphism between $BD(S,H)$ and $BD(S^{op},H^*)$

In [33], we saw that $Dys\text{-}_S\mathscr{D}^H = (Dys\text{-}_S\mathscr{D}^H, \tilde{\otimes}_S, S, \gamma)$, the category of dyslectic (S,H)-dimodules is a braided monoidal category and is an abelian full subcategory of $_S\mathscr{D}^H$ the category of (S,H)-dimodules.

Let us denote by $Dys\text{-}_S\mathscr{D}^{H\text{-}rev} = (Dys\text{-}_S\mathscr{D}^{H\text{-}rev}, \tilde{\otimes}_S^{rev}, S, \gamma^{rev})$ the reverse braided monoidal category of $Dys\text{-}_S\mathscr{D}^H$: we have $Dys\text{-}_S\mathscr{D}^{H\text{-}rev} = Dys\text{-}_S\mathscr{D}^H$ as a category with the same unit S, for two objects $M,N \in Dys\text{-}_S\mathscr{D}^{H\text{-}rev}$, $M\tilde{\otimes}_S^{rev}N = N\tilde{\otimes}_S M$, for two morphisms f and g in $Dys\text{-}_S\mathscr{D}^{H\text{-}rev}$, $f\tilde{\otimes}_S^{rev}g = g\tilde{\otimes}_S f$, and the braiding $\gamma_{M,N}^{rev}$ is equal to $\gamma_{N,M}$.

All the results established in [33] leading to the definition of the Brauer group $BD(S,H) = Br(Dys\text{-}_S\mathscr{D}^H)$ for the category $Dys\text{-}_S\mathscr{D}^H$ are also valid for the category $Dys\text{-}_S\mathscr{D}^{H\text{-}rev}$ and lead to the Brauer group $Br(Dys\text{-}_S\mathscr{D}^{H\text{-}rev})$ and we have:

$$Br(Dys\text{-}_S\mathscr{D}^{H\text{-}rev}) \cong BD(S,H)^{op}. \tag{3.3.1}$$

Now let's consider the braided monoidal category $Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$, the category consisting of dyslectic (S^{op},H^*)-dimodules and dyslectic (S^{op},H^*)-dimodule homomorphisms, that is, the left $S^{op}\#H^*$-linear right H^*-colinear maps. With this category, we obtain the Brauer group denoted $BD(S^{op},H^*)$ of dyslectic (S^{op},H^*)-dimodule Azumaya algebras.

Let M and N be two objects of $Dys\text{-}_S\mathscr{D}^H$. We define the map $\varphi_0 : S^{op} \to \mathscr{F}(S) = S$, it is the unit and a family of maps

$$\varphi_2(M,N) : \mathscr{F}(M)\tilde{\otimes}_{S^{op}}\mathscr{F}(N) = M\tilde{\otimes}_{S^{op}}N \to \mathscr{F}(M\tilde{\otimes}_S^{rev}N) = N\tilde{\otimes}_S M; \quad m\tilde{\otimes}_{S^{op}}n \mapsto n\tilde{\otimes}_S m.$$

Theorem 3.3.1. *The functor*

$$(\mathscr{F},\varphi_0,\varphi_2) : Dys\text{-}_S\mathscr{D}^{H\text{-}rev} \to Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$$

is an isomorphism of braided monoidal categories. Consequently

$$BD(S,H)^{op} \cong BD(S^{op},H^*),$$

as isomorphism of groups. This means that $BD(S,H)$ and $BD(S^{op},H^*)$ are anti-isomorphics Brauer-Clifford-Long groups.

Proof. For $s \in S, m \in M$ and $n \in N$,

$$\begin{aligned}
\varphi_2(M,N)((m \triangleleft s^o)\tilde{\otimes}_{S^{op}}n) &= \varphi_2(M,N)((m \triangleleft s^o)\tilde{\otimes}_{S^{op}}n)\\
&= \varphi_2(M,N)((s \rightharpoonup m)\tilde{\otimes}_{S^{op}}n)\\
&= n\tilde{\otimes}_S(s \rightharpoonup m)\\
&= (n \leftharpoonup s)\tilde{\otimes}_S m\\
&= \varphi_2(M,N)(m\tilde{\otimes}_{S^{op}}(n \leftharpoonup s))\\
&= \varphi_2(M,N)(m\tilde{\otimes}_{S^{op}}(s^o \triangleright n));
\end{aligned}$$

then $\varphi_2(M,N)$ is well-defined.

$$\begin{aligned}
\varphi_2(M,N)[s^o \triangleright (m\tilde{\otimes}_{S^{op}}n)] &= \varphi_2(M,N)[(s^o \triangleright m)\tilde{\otimes}_{S^{op}}n]\\
&= \varphi_2(M,N)[(m \leftharpoonup s)\tilde{\otimes}_{S^{op}}n]\\
&= n\tilde{\otimes}_S(m \leftharpoonup s)\\
&= (n\tilde{\otimes}_S m) \leftharpoonup s\\
&= s^o \triangleright (n\tilde{\otimes}_S m)\\
&= s^o \triangleright \varphi_2(M,N)(m\tilde{\otimes}_{S^{op}}n)
\end{aligned}$$

$$\begin{aligned}\varphi_2(M,N)(h^*(m\tilde{\otimes}_{S^{op}}n)) &= \varphi_2(M,N)((h_1^*m)\tilde{\otimes}_{S^{op}}(h_2^*n))\\ &= (h_2^*n)\tilde{\otimes}_S(h_1^*m)\\ &= (h_1^*n)\tilde{\otimes}_S(h_2^*m)\\ &= h^*(n\tilde{\otimes}_S m)\\ &= h^*\varphi_2(M,N)(m\tilde{\otimes}_{S^{op}}n)\end{aligned}$$

$$\begin{aligned}[\varphi_2(M,N)(m\tilde{\otimes}_{S^{op}}n)]_{(0)}\otimes[\varphi_2(M,N)(m\tilde{\otimes}_{S^{op}}n)]^*_{(1)} &= (n\tilde{\otimes}_S m)_{(0)}\otimes(n\tilde{\otimes}_S m)^*_{(1)}\\ &= (n_{(0)}\tilde{\otimes}_S m_{(0)})\otimes(n^*_{(1)}m^*_{(1)})\\ &= \varphi_2(M,N)(m_{(0)}\tilde{\otimes}_{S^{op}}n_{(0)})\otimes(m^*_{(1)}n^*_{(1)})\\ &= \varphi_2(M,N)((m\tilde{\otimes}_{S^{op}}n)_{(0)})\otimes(m\tilde{\otimes}_{S^{op}}n)^*_{(1)}\end{aligned}$$

that is, $\varphi_2(M,N)$ is left S^{op}-linear, H^*-linear and H^*-colinear. It is clear that $\varphi_2(M,N)$ is an isomorphism. According to [37, Definition. XI.4.1], $(\mathscr{F},\varphi_0,\varphi_2)$ is a monoidal functor. Now let's show that the funtor $(\mathscr{F},\varphi_0,\varphi_2)$ preserves the braiding. This means that the following diagram commutes

$$\begin{tikzcd} M\tilde{\otimes}_{S^{op}}N \arrow[r, "\gamma^*_{M,N}"] \arrow[d, "\varphi_2(M,N)"'] & N\tilde{\otimes}_{S^{op}}M \arrow[d, "\varphi_2(N,M)"] \\ N\tilde{\otimes}_S M \arrow[r, "\gamma_{N,M}=\gamma^{rev}_{M,N}"'] & M\tilde{\otimes}_S N \end{tikzcd}$$

Let $m\in M$ and $n\in N$, we have

$$\begin{aligned}[\varphi_2(N,M)\circ\gamma^*_{M,N}](m\tilde{\otimes}_{S^{op}}n) &= \varphi_2(N,M)[(m^*_{(1)}n)\tilde{\otimes}_{S^{op}}m_{(0)}]\\ &= m_{(0)}\tilde{\otimes}_S(m^*_{(1)}n)\\ &\overset{(3.1.16)}{=} m_{(0)}\tilde{\otimes}_S(n_0m^*_{(1)}(n_1))\\ &= (m_{(0)}m^*_{(1)}(n_1))\tilde{\otimes}_S n_0\\ &\overset{(3.1.18)}{=} (n_1m)\tilde{\otimes}_S n_0\\ &= \gamma_{N,M}(n\tilde{\otimes}_S m)\\ &= \gamma_{N,M}[\varphi_2(M,N)(m\tilde{\otimes}_{S^{op}}n)]\\ &= [\gamma_{N,M}\circ\varphi_2(M,N)](m\tilde{\otimes}_{S^{op}}n)\\ &= [\gamma^{rev}_{M,N}\circ\varphi_2(M,N)](m\tilde{\otimes}_{S^{op}}n)\end{aligned}$$

$\varphi_2(N,M)\circ\gamma^*_{M,N}=\gamma^{rev}_{M,N}\circ\varphi_2(M,N)$ then the diagram commutes, that is $\mathscr{F}$ preserves the braiding. It is clear that $\mathscr{F}: Dys\text{-}_S\mathscr{D}^{H\text{-}rev}\to Dys\text{-}_{S^{op}}\mathscr{D}^{H^*}$ is an isomorphism of functors. Therefore $(\mathscr{F},\varphi_0,\varphi_2)$ is a braided monoidal functor and we immediately have

$$Br(Dys\text{-}_S\mathscr{D}^{H\text{-}rev})\cong BD(S^{op},H^*), \tag{3.3.2}$$

as isomorphism of groups. Finally we have

$$BD(S,H)^{op}\overset{(3.3.1)}{\cong}Br(Dys\text{-}_S\mathscr{D}^{H\text{-}rev})\overset{(3.3.2)}{\cong}BD(S^{op},H^*)$$

□

Remark 3.3.2. *In [33], we have shown that, for a commutative cocommutative Hopf algebra H and an H-commutative H-dimodule algebra S, there is an anti-isomorphism of groups $BD(S,H) \cong BQ(S^{op},H)$ mapping $[A]$ to $[[A^{op}]]$, where $BQ(S^{op},H)$ is the Brauer group of dyslectic Hopf Yetter-Drinfel'd (S^{op},H)-module Azumaya algebras. Therefore if H is commutative and cocommutative finitely generated projective Hopf algebra and S a H-commutative H-dimodule algebra, we have $BQ(S^{op},H) \cong BD(S^{op},H^*)$ as isomorphism of groups.*

In many applications, H could be the group algebra RG of a finite abelian group G (for example the cyclic group of order 2) or the set $Maps(G,\mathbb{k})$ of all maps from G to $\mathbb{k}$, where G is a finite abelian group and $\mathbb{k}$ is a field.

Chapter 4

ROSENBERG-ZELINSKY EXACT SEQUENCE

abstract

In this chapter, we generalize the Rosenberg-Zelinsky sequence to dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebras which terms are the group of all H-inner (H-INNER) S-algebra automorphisms of an algebra $A \in Dys\text{-}_S\mathcal{Q}^H$ and the group of isomorphism classes of invertible S-modules (invertible dyslectic Hopf Yetter-Drinfeld (S,H)-modules) under the tensor product $\tilde{\otimes}_S$ denoted $Pic(S)$ $(P\mathcal{Q}^H(S,H))$. When H is a commutative cocommutative Hopf algebra, the dyslectic (T,H)-dimodule version of the Rosenberg-Zelinsky exact sequence is also given.

Introduction

Throughout the paper, R will denote a commutative ring with 1. All of the R-algebras and R-modules we will consider here are symmetric R-R-bimodules. A right (left) R-module is faithfully projective if it is finitely generated projective and faithfull as a right (left) R-module. All unadorned tensor products and sets of module homomorphisms are intended to be over R and will be specified if otherwise.

In their study to show that for any commutative ring R (here $C = R$) and for any simple central R-algebra A, $\mathscr{J}(R)$ (the abelian group under the operation $\otimes$ of isomorphism classes of finitely generated, projective R-modules of rank one) contains a subgroup which is isomorphic to a group of automorphisms modulo the inner ones in [51], Rosenberg and Zelinsky established and proved that the sequence

$$1 \longrightarrow Aut(A) \xrightarrow{\alpha} \mathscr{J}(R) \xrightarrow{\beta} \mathscr{J}(A) \longrightarrow 1$$

is an exact sequence, were A is a central simple R-algebra, $Aut(A)$ denotes the abelian group of automorphisms of A modulo the inner ones and $\mathscr{J}(A)$ is the set of left A-isomorphism classes of left $A \otimes A^{op}$-modules which are finitely generated and projective as R-modules. This sequence is called in the literature the ***Rosenberg-Zelinsky exact sequence***. Subsequently several versions of this sequence have been defined for other spaces for example [21, Theorem 3.1], [13], [17, Proposition 3.5], [14, Theorem 13.6.1], [39, Section 4.1], [10]...

Let A be a Yetter-Drinfeld H-module algebra. Denote by H-$Aut(A)$ the group of all Yetter-Drinfeld H-module algebra automorphisms of A. An H-automorphism f of A is called H-INNER if there exists an invertible element $x \in A$ such that $f(a) = xax^{-1}$ for all $a \in A$. The subgroup of H-$Aut(A)$ consisting of H-INNER automorphisms of A will be denoted by H-$INN(A)$. $f \in H$-$Aut(A)$ is called H-inner if there exist an invertible element $x \in$ A and a grouplike element $g \in G(H)$ (the set of grouplike elements ot H) such that $f(a) = x(g.a)x^{-1}$ for all $a \in A$. Denote by $PQ(R,H)$ the group of isomorphism classes of invertible Yetter-Drinfeld H-modules under the tensor product and called the Picard group of the Hopf algebra H. For an H-Azumaya algebra A, Caenepeel, Oystaeyen and Zhang [17, Proposition 3.5] give the generalized Rosenberg-Zelinsky exact sequence

$$1 \longrightarrow H\text{-}INN(A) \longrightarrow H\text{-}Aut(A) \xrightarrow{\Phi} PQ(k,H)$$

where

$$\Phi(f) = I_f = \{x \in A \mid \sum x_0(x_1.a) = f(a), \quad \forall x \in A\}$$

Now let H be a commutative cocommutative finitely generated and projective Hopf algebra over R and A an H-Azumaya H-dimodule algebra. Let H-$Aut(A)$ be the group of all H-linear, H-colinear R-automorphisms of A. In [14, Theorem 13.6.1], Caenepeel established a *Dimodule version of the Rosenberg-Zelinsky exact sequence*

$$1 \longrightarrow H\text{-}INN(A) \longrightarrow H\text{-}Aut(A) \xrightarrow{\Phi} PD(R,H)$$

and

$$1 \longrightarrow H\text{-}Inn(A) \longrightarrow H\text{-}Aut(A) \xrightarrow{\Psi} Pic(R),$$

where H-$Inn(A)$ is the group of inner automorphisms of A that is the set of elements $\alpha \in H$-$Aut(A)$ for which there exists an invertible $u \in A$ such that

$$\alpha(x) = \sum (x_1.u)x_0u^{-1}, \quad \forall x \in A.$$

If the action and the coaction of H on $u \in A$ are all trivial, $\alpha \in H$-$Inn(A)$ is called H-$INNER$ and H-$INN(A)$ denotes the subgroup of H-$Aut(A)$ consiting of H-$INNER$ automorphisms of A. $Pic(R)$ denotes the group of isomorphism classes of invertible R-modules called the Picard group of R and $PD(R,H)$ the group of isomorphism classes of invertible H-dimodules algebras under the tensor product $\otimes$, called the Picard group of H-dimodules algebras.

In this chapter, our first aim is to give a version of the Rosenberg-Zelinsky exact sequences for the dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebras (where S is an H-commutative Hopf Yetter-Drinfel'd H-module algebra, see [32]) and the dyslectic (S,H)-dimodule Azumaya algebras (where S is an H-commutative H-dimodule algebra and H a commutative, cocommutative Hopf algebra, see [33]).

Our work is carried out according to the following plan: in the Section 4.1, we have given the preliminary notions of Hopf algebras and recalled the results of [32] for the dyslectic Hopf Yetter-Drinfel'd (S,H)-module adding some small results and ingredients which are useful for the following sections. In the Section 4.2 we have established the Morita context for the category $Dys\text{-}_S\mathcal{Q}^H$ of the dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebras, before giving in Section 4.3 the main results of this paper, namely the Rosenberg-Zelinsky exact sequence for the category $Dys\text{-}_S\mathcal{Q}^H$. If H is commutative and cocommutative Hopf algebra, then Yetter-Drinfel'd H-modules become H-dimodules. Adopting the results of [33], taking into account to the braiding of H-dimodules in Section 4.4 to establish the sequence for the category $Dys\text{-}_S\mathcal{D}^H$ of dyslectic (S,H)-dimodule Azumaya algebras.

4.1 Dyslectic Hopf Yetter-Drinfeld (S,H)-module algebras

For background on Hopf algebras and coactions of Hopf algebras on rings, we refer the reader to [56] and [44]. Let H be a Hopf R-algebra with comultiplication $\Delta_H : H \to H \otimes H$, antipode $S_H : H \to H$ and counit $\epsilon_H : H \to R$. We will use Sweedler-Heyneman notation but we will omit the symbol $\sum$:

$$\Delta_H(h) = h_1 \otimes h_2, \quad \forall h \in H.$$

We say that an R-algebra A is an *H-module algebra* if A is a left H-module such that

$$h.(ab) = (h_1.a)(h_2.b) \quad \text{and} \quad h.1_A = \epsilon_H(h)a, \quad \forall \quad a,b \in A, h \in H. \tag{4.1.1}$$

A homomorphism of H-module algebras is a homomorphism of H-modules which is also a homomorphism of R-algebras. H is said to act trivially on A when $h.a = \epsilon_H(h)a$ for all $h \in H$ and $a \in A$.

If S is an H-module algebra, then the smash product algebra $S\#H$ is the R-module $S \otimes H$ endowed with the product

$$(s \otimes h)(s' \otimes h') = s(h_1.s') \otimes h_2 h', \text{for all } s,s' \in S \text{ and } h,h' \in H. \tag{4.1.2}$$

An R-module M is a left $S\#H$-module if it is a left S-module and a left H-module for which

$$h.(sm) = (h_1.s)(h_2 m) \tag{4.1.3}$$

for all $h \in H$, $s \in S$ and $m \in M$. If A is an H-module algebra and S is a sub-H-module algebra of A, then the algebras A and S are left $S\#H$-modules.

Let us denote by ${}_{S\#H}\mathcal{M}$ the category of $S\#H$-modules.

Let H be a Hopf algebra. An R-module M is a right H-comodule if there exists an R-linear map $\rho_M : M \to M \otimes H$ satisfying

$$(\rho_M \otimes id_H) \circ \rho_M = (id_M \otimes \Delta_H) \circ \rho_M \quad \text{and} \quad (id_M \otimes \epsilon_H) \circ \rho_M = id_M.$$

We write $\rho_M(m) = m_0 \otimes m_1$ for $m \in M$.

By [44, Example 1.6.7] or [40, Proposition 2.2], if $H = kG$ is a group algebra, then M is a right H-comodule if and only if M is a left G-graded module.

Let M and N be right H-comodules. A homomorphism of right H-comodules or a right H-colinear map $M \to N$ is an R-linear map $f : M \to N$ such that $\rho_N \circ f = (f \otimes id_H) \circ \rho_M$; in Sweedler's notation, this is equivalent to

$$f(m)_0 \otimes f(m)_1 = f(m_0) \otimes m_1 \tag{4.1.4}$$

If M and N are right H-comodules, then $M \otimes N$ is a right H-comodule under the codiagonal coaction; that is,

$$\rho(m \otimes n) = (m_0 \otimes n_0) \otimes (m_1 n_1), \quad \forall m \in M, n \in N \tag{4.1.5}$$

If M is finitely generated projective as an R-module or if H is finitely generated projective as an R-module, then we have a natural R-isomorphism $Hom(M,N \otimes H) \cong Hom(M,N) \otimes H$ (see [12, Proposition 2 or Proposition 4, Chapter 2]). Using this isomorphism, we define an R-linear map

$$\rho : Hom(M,N) \longrightarrow Hom(M,N) \otimes H; \;\; f \longmapsto f_0 \otimes f_1$$

by

$$\rho(f)(m) = f_0(m) \otimes f_1 = f(m_0)_0 \otimes [S_H^{-1}(m_1) f(m_0)_1] \tag{4.1.6}$$

defining a right H-comodule structure on $Hom(M,N)$ if M is finitely generated and projective as an R-module or if H is finitely generated projective as an R-module.

In this paper, $Hom(M,N)$ and $M \otimes N$ (if M and N are right H-comodules that are finitely generated projective as R-modules) will be considered as H-comodules with the H-coaction defined above unless explicitly stated otherwise.

We say that an R-algebra A is an *H-comodule algebra* if A is a right H-comodule such that

$$\rho(ab) = (a_0 b_0) \otimes (a_1 b_1), \quad \text{and} \quad \rho(1_A) = 1_A \otimes 1_H \quad \forall a, b \in A. \tag{4.1.7}$$

A homomorphism of H-comodule algebras is a homomorphism of H-comodules which is also a homomorphism of R-algebras. The coaction of H on A is trivial when $\rho(a) = a \otimes 1_H$ for all $a \in A$.

Let S be a right H-comodule algebra. An R-module M is an (S,H)-Hopf module if M is a left S-module and a right H-comodule such that

$$(sm)_0 \otimes (sm)_1 = (s_0 m_0) \otimes (s_1 m_1). \tag{4.1.8}$$

A homomorphism of (S,H)-Hopf modules is a left S-linear right H-colinear map.

Definition 4.1.1. *Let H be a Hopf algebra with bijective antipode. A left Yetter-Drinfeld H-module M is an R-module with an H-action and an H-coaction such that the following compatibility relation holds, for all $m \in M$ and $h \in H$;*

$$\rho_M(hm) = (hm)_0 \otimes (hm)_1 = (h_2 m_0) \otimes (h_3 m_1 S_H^{-1}(h_1)) \tag{4.1.9}$$

Yetter-Drinfeld modules are sometimes called *crossed modules,* or *Quantum Yang-Baxter modules*. If H is commutative and cocommutative , then the relation (4.1.9) becomes (2.1.9) that is:

$$\rho_M(hm) = (hm_0) \otimes m_1 \tag{4.1.10}$$

and we see that Yetter-Drinfeld modules generalize dimodules.

Definition 4.1.2. *A Hopf Yetter-Drinfeld H-module algebra is an R-algebra which is a Hopf Yetter-Drinfeld H-module so that it is a left H-module algebra and a right H^{op}-comodule algebra satisfying the relation (4.1.9).*

A *Hopf Yetter-Drinfeld H-module homomorphism* between two Hopf Yetter-Drinfeld H-modules M and N is an R-linear homomorphism $M \to N$ which is simultaneously an H-module homomorphism and an H^{op}-comodule homomorphism. A *Hopf Yetter-Drinfeld H-module algebra homomorphism* between two Hopf Yetter-Drinfeld H-module algebras A and B is an R-linear map $A \to B$ which is simultaneously a Hopf Yetter-Drinfeld H-module homomorphism and an R-algebra homomorphism. If H is commutative and cocommutative, then a Yetter-Drinfeld module algebra is nothing else then an H-dimodule algebra.

The tensor product of two Yetter-Drinfeld modules M and N is again a Yetter-Drinfeld module. The H-action on $M \otimes N$ is the diagonal action

$$h(m \otimes n) = (h_1 m) \otimes (h_2 n), \tag{4.1.11}$$

for all $h \in H, m \in M, n \in N$, and the H-coaction is given by the codiagonal coaction (formula (4.1.5)). The category of Yetter-Drinfeld H-modules and H-linear H^{op}-colinear maps is denoted by $\mathcal{Q}^H$.

In the sense of Maclane (see [42]), the category $(\mathcal{Q}^H, \otimes, R)$ is a monoidal category.

The monoidal category $(\mathcal{C}, \otimes)$ is *braided* if for all M, N and $P \in \mathcal{C}$ there exists a natural isomorphism $\gamma_{M,N} : M \otimes N \to N \otimes M$ such that the following conditions are satistafied (see [44, p.198])

$$\gamma_{M\otimes P,N} = (\gamma_{M,P} \otimes I_N) \circ (I_M \otimes \gamma_{N,P}) \quad \text{and} \quad \gamma_{M,N\otimes P} = (I_N \otimes \gamma_{M,P}) \circ (\gamma_{M,N} \otimes I_P).$$

For all objects M and N in $\mathcal{Q}^H$, the map

$$\gamma_{M,N} : M \otimes N \to N \otimes M;\ \ m \otimes n \mapsto n_0 \otimes (n_1 m),$$

$\forall m \in M, n \in N$, is an isomorphism in $\mathcal{Q}^H$ for which the inverse $\gamma_{M,N}^{-1}$ from $N \otimes M$ to $M \otimes N$, is given by

$$\gamma_{M,N}^{-1}(n \otimes m) = (S_H(n_1)m) \otimes n_0.$$

By [18] and [17], $(\mathcal{Q}^H, \otimes, R, \gamma_{M,N})$ is a braided monoidal category.

Definition 4.1.3. *Let S be a Hopf Yetter-Drinfeld H-module algebra. A Hopf Yetter-Drinfeld (S,H)-module is an S-module and a Hopf Yetter-Drinfeld H-module M such that the H-action and the H-coaction commute with the S-action, or equivalently, M is a left $S\#H$-module and a right (S,H^{op})-Hopf module (with the same S-action) for which the relation (4.1.9) is satisfied.*

If S is a Hopf Yetter-Drinfeld H-module algebra, then S is a Hopf Yetter-Drinfeld (S,H)-module. A Hopf Yetter-Drinfeld (R,H)-module is just a Hopf Yetter-Drinfeld H-module.

A Hopf Yetter-Drinfeld (S,H)-module map between two Hopf Yetter-Drinfeld (S,H)-modules M and N is an S-linear homomorphism $M \to N$ which is also a Hopf Yetter-Drinfeld H-module map. We denote by ${}_S\mathscr{Q}^H$ the category of Hopf Yetter-Drinfeld (S,H)-modules: its morphisms are the Hopf Yetter-Drinfeld (S,H)-module maps.

Definition 4.1.4. *A Hopf Yetter-Drinfeld H-module algebra S is said to be H-commutative (or quantum commutative) if*

$$st = t_0(t_1.s), \quad \forall s,t \in S. \tag{4.1.12}$$

If S is defined as above, we can define the corresponding right S-action and the left S-action on $M \in {}_S\mathscr{Q}^H$ by

$$ms = s_0(s_1m) \tag{4.1.13}$$

and

$$sm = (S_H(s_1)m)s_0, \tag{4.1.14}$$

for all $m \in M, s \in S$. From this, M becomes an S-S-bimodule.

If S is an H-commutative Yetter-Drinfeld H-module algebra, the left H-action and the right S-action on M are compatible, that is

$$h(ms) = (h_1m)(h_2.s), \quad \forall h \in H, m \in M, s \in S \tag{4.1.15}$$

and the right H-coaction is also compatible with the right S-action on M, that is,

$$(ms)_0 \otimes (ms)_1 = (m_0s_0) \otimes (s_1m_1) \quad \forall m \in M, s \in S. \tag{4.1.16}$$

Let S be an H-commutative Hopf Yetter-Drinfeld H-module algebra and $\otimes_S$ denotes tensor product over S. If M and N are objects of ${}_S\mathscr{Q}^H$, then, so is $M \otimes_S N$ (see [17]) where the left S-action, the left H-action and the right H-coaction on $M \otimes_S N$ are given by the formulas

$$s(m \otimes_S n) = (sm) \otimes_S n, \tag{4.1.17}$$

$$h(m \otimes_S n) = (h_1m) \otimes_S (h_2n) \tag{4.1.18}$$

$$(m \otimes_S n)_0 \otimes (m \otimes_S n)_1 = (m_0 \otimes_S n_0) \otimes (n_1m_1) \tag{4.1.19}$$

for all $s \in S, h \in H, m \in M$ and $n \in N$. The corresponding right S-action on $M \otimes_S N$ is given by

$$(m \otimes_S n)s = m \otimes_S (ns). \tag{4.1.20}$$

According to [18] the category $({}_S\mathscr{Q}^H, \otimes_S, S)$ is a monoidal category.

Let S be an H-commutative Hopf Yetter-Drinfeld H-module algebra and M and N two Hopf Yetter-Drinfeld (S,H)-modules, in this part $Hom_S(M,N)$ is the set of right S-linear maps from M to N and ${}_SHom(M,N)$ the set of left S-linear maps. According to [32, Lemma 2.1]

$Hom_S(M,N)$ is a left $S\#H$-module, where the left S-action and H-action are respectively given by

$$(sf)(m) = sf(m) \tag{4.1.21}$$

and

$$(h.f)(m) = h_1[f(S_H(h_2)m)] \tag{4.1.22}$$

for all $s \in S, m \in M, n \in N, h \in H$ and $f \in Hom_S(M,N)$. If M is finitely generated projective as a right S-module, then $Hom_S(M,N)$ is a Hopf Yetter-Drinfeld (S,H)-module where the H-coaction is defined by formula (4.1.6). By [32, Lemma 2.3], we have an isomorphism of R-modules

$$_{S\#H}Hom^{H^{op}}(M \otimes_S N, P) \cong {}_{S\#H}Hom^{H^{op}}(M, Hom_S(N,P)).$$

Similarly, from [32, Lemma 2.2] $_SHom(M,N)$ is a left $S\#H$-module where the left S-action and H-action are respectively given by

$$(sf)(m) = f(ms) \tag{4.1.23}$$

and

$$(h.f)(m) = h_2[f(S_H^{-1}(h_1)m)] \tag{4.1.24}$$

for all $s \in S, m \in M, n \in N, h \in H$ and $f \in {}_SHom(M,N)$. If M is finitely generated and projective as a right S-module, then $_SHom(M,N)$ is a Hopf Yetter-Drinfeld (S,H)-module where the H-coaction is defined by

$$f(m)_0 \otimes f_1 = f(m_0)_0 \otimes [f(m_0)_1 S_H(m_1)]. \tag{4.1.25}$$

By [32, Lemma 2.4], we have an isomorphism of R-module

$$_{S\#H}Hom^{H^{op}}(M \otimes_S N, P) \cong {}_{S\#H}Hom^{H^{op}}(M, {}_SHom(N,P)).$$

We deduce from [32, Lemma 2.3], that if P finitely generated and projective as a right S-module, then

$$\begin{cases} Hom_S(P,\bullet) : {}_S\mathcal{Q}^H \longrightarrow {}_S\mathcal{Q}^H \\ \\ \bullet \otimes_S P : {}_S\mathcal{Q}^H \longrightarrow {}_S\mathcal{Q}^H \end{cases}$$

is a pair of adjoint functors.

Similarly, we deduce from [32, Lemma 2.4], that if P finitely generated and projective as a left S-module, then

$$\begin{cases} {}_SHom(P,\bullet) : {}_S\mathcal{Q}^H \longrightarrow {}_S\mathcal{Q}^H \\ \\ P \otimes_S \bullet : {}_S\mathcal{Q}^H \longrightarrow {}_S\mathcal{Q}^H \end{cases}$$

is pair of adjoint functors.

Definition 4.1.5. *A dyslectic Hopf Yetter-Drinfeld (S,H)-module M is a Hopf Yetter-Drinfeld (S,H)-module such that $\alpha_M \circ \gamma_{M,S} \circ \gamma_{S,M} = \alpha_M$, where α_M denotes the left S-action on M. In other words, M is dyslectic if*

$$sm = m_0(m_1.s), \quad \forall m \in M, s \in S, \tag{4.1.26}$$

its right equivalent condition is

$$ms = (S_H(m_1).s)m_0 \tag{4.1.27}$$

Note that S is a dyslectic Hopf Yetter-Drinfeld (S,H)-module, and every Hopf YetterDrinfeld H-module can be regarded as a dyslectic Hopf Yetter-Drinfeld (R,H)-module. A Hopf Yetter-Drinfeld (S,H)-module homomorphism between dyslectic Hopf Yetter-Drinfeld (S,H)-modules is called *dyslectic Hopf Yetter-Drinfeld (S,H)-module homomorphism*. The category of dyslectic Hopf Yetter-Drinfeld (S,H)-modules with dyslectic Hopf Yetter-Drinfeld (S,H)-modules homomorphisms is denoted by $Dys\text{-}{}_S\mathscr{Q}^H$. It is a full subcategory of ${}_S\mathscr{Q}^H$.

According to [32, Lemma 3.4] for M and N two dyslectic Hopf Yetter-Drinfeld (S,H)-modules, $M \otimes_S N$ is a dyslectic Hopf Yetter-Drinfeld (S,H)-module. It follows that, $(Dys\text{-}{}_S\mathscr{Q}^H, \otimes_S, S, \gamma)$ is a braided monoidal category.

For M and N two dyslectic Hopf Yetter-Drinfeld (S,H)-modules, if M is finitely generated and projective as a right S-module, by [32, Lemma 3.5], $Hom_S(M,N)$ and ${}_SHom(M,N)$ are dyslectic Hopf Yetter-Drinfeld (S,H)-modules. Therefore, the functors

$$\begin{cases} Hom_S(P,\bullet) : Dys\text{-}{}_S\mathscr{Q}^H \longrightarrow Dys\text{-}{}_S\mathscr{Q}^H \\ \\ \bullet \otimes_S P : Dys\text{-}{}_S\mathscr{Q}^H \longrightarrow Dys\text{-}{}_S\mathscr{Q}^H \end{cases} \text{and} \begin{cases} {}_SHom(P,\bullet) : Dys\text{-}{}_S\mathscr{Q}^H \longrightarrow Dys\text{-}{}_S\mathscr{Q}^H \\ \\ P \otimes_S \bullet : Dys\text{-}{}_S\mathscr{Q}^H \longrightarrow Dys\text{-}{}_S\mathscr{Q}^H \end{cases}$$

are two pairs of adjoint functors (where P is finitely generated projective as a right S-module).

Since the category $Dys\text{-}{}_S\mathscr{Q}^H$ is braided, for two dyslectic Hopf Yetter-Drinfeld (S,H)-modules P and Q, the map

$$\phi : Hom_S(P,Q) \to {}_SHom(P,Q) \quad \text{by} \quad \phi(f)(p) = f_0(f_1p),$$

where P is finitetely generated projective as right S-module, is an isomorphism of dyslectic Hopf Yetter-Drinfeld (S,H)-modules (see [32]).

A *dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra* is an algebra in the braided monoidal category $Dys\text{-}{}_S\mathscr{Q}^H$. A *dyslectic Hopf Yetter Drinfeld (S,H)-module algebra homomorphism* is a dyslectic Hopf Yetter-Drinfeld (S,H)-module homomorphism which is compatible with the product and is a unitary algebra homomorphism. By [32, Lemma 4.1 and 4.2], if M is a dyslectic Hopf Yetter-Drinfeld (S,H)-module that is finitely generated projective as a right (left) S-module, $End_S(M)$ (${}_SEnd(M)$) is a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra.

Let A be a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra. The H-opposite algebra $\bar{A}$ of A ([59, page 100]) is defined as follows: $\bar{A} = A$ as a dyslectic Hopf Yetter-Drinfeld (S,H)-module, but with multiplication $m_A \circ \gamma$, where m_A is the multiplication of A. In other words,

$$\bar{a}.\bar{a}' = \overline{a'_0(a'_1.a)}, \quad \forall a,a' \in A. \tag{4.1.28}$$

Note that the action of S on $\bar{A}$ is defined by $s.\bar{a} = \overline{sa}$, the H-action and the H-coaction are respectively $h.\bar{a} = \overline{h.a}$ and $(\bar{a})_0 \otimes (\bar{a})_1 = \overline{a_0} \otimes a_1$. If the action of H or the coaction of H is trivial, then $\bar{A} = A^{op}$, the opposite algebra of A. Note that $\bar{S} = S$: which means that $\bar{S}$ is H-commutative (or quantum commutative) as an algebra in the category $Dys\text{-}_S\mathcal{Q}^H$.

Lemma 4.1.6. *Suppose that A is a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra. Then*

i) A^{op} is a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra: the S-action is defined by $s.a^o = (s.a)^o$, the H-action is defined by $h.a^o = (h.a)^o$ and the H-coaction is defined by

$$(a^o)_0 \otimes (a^o)_1 = (a_0)^o \otimes a_1$$

ii) $\bar{A}$ is a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra (see [32, Lemma 4.3]).

If M is an S-module, we set $Hom_S(M,S) = M^*$ the right dual of M. It is a dyslectic Hopf Yetter-Drinfeld (S,H)-module.

A dyslectic Hopf Yetter-Drinfeld (S,H)-module is right faithfully projective if it is finitely generated projective as a right S-module and the canonical map

$$\psi : Hom_S(P,S) \otimes_{End_S(P)} P \to S; \;\; f \tilde{\otimes} p \mapsto f(p)$$

is an isomorphism.

We define in a similar way a left faithfully projective dyslectic Hopf Yetter-Drinfeld (S,H)-module. A dyslectic Hopf Yetter-Drinfeld (S,H)-module is said to be *faithfully projective* if it is right and left faithfully projective.

Since $\mathcal{D}ys\text{-}_S\mathcal{Q}^H$ is a braided monoidal category, by [23], a dyslectic Hopf Yetter-Drinfeld (S,H)-module is right faithfully projective if and only if it is left faithfully projective. So a dyslectic Hopf Yetter-Drinfeld (S,H)-module is faithfully projective if it is right faithfully projective or left faithfully projective.

If A and B are dyslectic Hopf Yetter-Drinfeld (S,H)-module algebras, we define a new multiplication in $A \otimes_S B$ by

$$(A \otimes_S B) \otimes_S (A \otimes_S B) \overset{I_A \otimes \gamma \otimes I_B}{\to} (A \otimes_S A) \otimes_S (B \otimes_S B) \overset{m_A \otimes m_B}{\to} A \otimes_S B.$$

In other words,

$$(a \otimes b)(a' \otimes b') = (aa'_0) \otimes (a'_1.b)b', \quad \forall a,a' \in A \text{ and } \forall b,b' \in B. \tag{4.1.29}$$

This new multiplication on $A\otimes_S B$ is called the braided product. The dyslectic Hopf Yetter-Drinfeld (S,H)-modules $A\otimes_S B$ with the braided product is usually denoted $A\otimes_\gamma B$. It is denoted $A\#_\gamma B$ in [16] and called generalized smash product. We will denote it by $A\#_S B$ in this paper. For more details and results about this product, we refer to [32].

Let A be a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra which is faithfully projective as an S-module. We define the S-linear maps

$$F : A\#_S\bar{A} \to End_S(A) \quad \text{given by} \quad (a\#\bar{b})(c) = ac_0(c_1.b)$$

$$G : \bar{A}\#_S A \to \overline{End_S(A)} \cong {}_SEnd(A) \quad \text{given by} \quad (\bar{a}\#b)(c) = a_0(a_1.c)b,$$

for all $a,b,c\in A$. According to [32, Proposition 5.1], the maps F and G are dyslectic Hopf Yetter-Drinfeld (S,H)-module homomorphisms.

Definition 4.1.7. *Let A be a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra which is faithfully projective as an S-module. We say that A is an Azumaya algebra in the category $Dys\text{-}_S\mathcal{Q}^H$ if A is faithfully projective as an S-module, and the S-linear maps F and G are isomorphisms of dyslectic Hopf Yetter-Drinfeld (S,H)-module algebras.*

An Azumaya algebra in $Dys\text{-}_S\mathcal{Q}^H$ will be called a dyslectic Hopf Yetter-Drinfeld (S,H)-module Azumaya algebra. If the H-coaction is trivial in Definition 4.1.7, then A is just an (S,H)-algebra which is faithfully projective as an S-module, such that the natural map $A\otimes_S A^{op}\to End_S(A)$ is an isomorphism of (S,H)-algebras and S is commutative. So A is an (S,H)-Azumaya algebra. In the same way, if the H-action is trivial in Definition 4.1.7, then A is just an (S,H)-algebra which is faithfully projective as an S-module, such that the natural map $A\otimes_S A^{op}\to End_S(A)$ is an isomorphism of (S,H)-Hopf algebras and S is commutative. So A is an (S,H)-Hopf Azumaya algebra.

4.2 Morita theory for dyslectic Hopf Yetter-Drinfeld (S,H)-modules

Throughout this section, H is a Hopf algebra with bijective antipode and S is an H-commutative dyslectic Hopf Yetter-Drinfeld H-module.

Definition 4.2.1. *Let A and B be dyslectic Hopf Yetter-Drinfeld (S,H)-module Azumaya algebras. A dyslectic Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule is an object of $Dys\text{-}_S\mathcal{Q}^H$ which is an $(A\text{-}B)$-bimodule such that the bimodule map $A\otimes_S M\otimes_S B\to M$ is compatible with the left H-action and the right H-coaction, that is, for all $a\in A, b\in B, h\in H$ and $m\in M$ we have;*

$$h(amb) = (h_1.a)(h_2m)(h_3.b) \tag{4.2.1}$$

and

$$(amb)_0\otimes(amb)_1 = (a_0m_0b_0)\otimes(a_1m_1b_1). \tag{4.2.2}$$

A dyslectic Hopf Yetter-Drinfeld $(S\text{-}S,H)$-bimodule is just a dyslectic Hopf Yetter-Drinfeld (S,H)-module. An *homomorphism of dyslectic Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule* is a dyslectic Hopf Yetter-Drinfeld (S,H)-module homomorphism which is $(A\text{-}B)$-bilinear (that is, which is simultaniously left A-linear and right B-linear).

We denote by $Dys\text{-}_A\mathcal{Q}_B^H$ the category of dyslectic Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule and dyslectic Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule homomorphisms.

According to [59], a dyslectic Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule is an $(A\text{-}B)$-bimodule in $Dys\text{-}_S\mathcal{Q}^H$. A dyslectic Hopf Yetter-Drinfeld $(A\text{-}S,H)$-bimodule is a left A-module in $Dys\text{-}_S\mathcal{Q}^H$. Similarly, a dyslectic Hopf Yetter-Drinfeld $(S\text{-}A,H)$-module is a right A-module in $Dys\text{-}_S\mathcal{Q}^H$. So a dyslectic Hopf Yetter-Drenfil'd $(S\text{-}S,H)$-bimodule is a left and right S-module in $Dys\text{-}_S\mathcal{Q}^H$.

A Hopf Yetter-Drinfel'd $(A\text{-}S,H)$-bimodule will be called left Yetter-Drinfel'd (A,H)-bimodule, likewise, a Hopf Yetter-Drinfel'd $(S\text{-}A,H)$-bimodule will be called right Yetter-Drinfel'd (A,H)-bimodule.

Consider two dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebras A and B.

Lemma 4.2.2. *Let P be a dyslectic Hopf Yetter-Drinfel'd $(A\text{-}B,H)$-bimodule and Q a dyslectic Hopf Yetter-Drinfeld $(B\text{-}A,H)$-bimodule. Then*

i) *$P \otimes_B Q$ is a Hopf Yetter-Drinfeld $(A\text{-}A,H)$-bimodule,*

ii) *$Q \otimes_A P$ is a Hopf Yetter-Drinfeld $(B\text{-}B,H)$-bimodule,*

iii) • *if P is a Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule finitely generated projective as a right B-module, then $Hom_B(P,B)$ is a Hopf Yetter-Drinfeld $(B\text{-}A,H)$-bimodule.*

Moreover, $End_B(P)$ is a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra.

• *if P is a Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule finitely generated projective as a left A-module, then ${}_AHom(P,A)$ is a Hopf Yetter-Drinfeld $(B\text{-}A,H)$-bimodule.*

Moreover, ${}_AEnd(P)$ is a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra.

iv) • *if Q is a Hopf Yetter-Drinfeld $(B\text{-}A,H)$-module finitely generated projective as a right A-module , then $Hom_A(Q,A)$ is a Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodules.*

Moreover, $End_A(Q)$ is a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra.

• *if Q is a Hopf Yetter-Drinfeld $(B\text{-}A,H)$-module finitely generated projective as a left B-module, then ${}_BHom(Q,B)$ is a Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule.*

Moreover, ${}_BEnd(Q)$ is a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra.

Proof.

i) Let $a,a' \in A$ and $p \otimes_B q \in P \otimes_B Q$,

$$(aa')(p \otimes_B q) = ((aa')p) \otimes_B q = (a(a'p)) \otimes_B q = a((a'p) \otimes_B q) = a(a'(p \otimes_B q)),$$

$$(p\otimes_B q)(a'a) = p\otimes_B (q(a'a)) = p\otimes_B ((qa')a) = (p\otimes_B (qa'))a = ((p\otimes_B q)a')a,$$

$$(a(p\otimes_B q))a' = ((ap)\otimes_B q)a' = (ap)\otimes_B (qa') = a(p\otimes_B (qa')) = a((p\otimes_B q)a')$$

$$1_A(p\otimes_B q) = (1_A p)\otimes_B q = p\otimes_B q \text{ and } (p\otimes_B q)1_A = p\otimes_B (q1_A) = p\otimes_B q,$$

then $P\otimes_B Q$ is an $(A\text{-}A)$-bimodule. We have:

$$\begin{aligned}
(p\otimes_B q)_0\otimes\Delta((p\otimes_B q)_1) &= (p_0\otimes_B q_0)\otimes\Delta(q_1p_1)\\
&= (p_0\otimes_B q_0)\otimes(q_1p_1)_1\otimes(q_1p_1)_2\\
&= (p_0\otimes_B q_0)\otimes(q_{11}p_{11})\otimes(q_{12}p_{12})\\
&= (p_{00}\otimes_B q_{00})\otimes(q_{01}p_{01})\otimes(q_1p_1)\\
&= (p_0\otimes_B q_0)_0\otimes(p_0\otimes_B p_0)_1\otimes(q_1p_1)\\
&= \rho(p_0\otimes_B q_0)\otimes(q_1p_1)\\
&= \rho((p\otimes_B q)_0)\otimes(p\otimes_B q)_1
\end{aligned}$$

$$(p\otimes_B q)_0\otimes\varepsilon((p\otimes_B q)_1) = (p_0\otimes_B q_0)\varepsilon(q_1p_1) = p_0\varepsilon(p_1)\otimes_B q\varepsilon(q_1) = p\otimes_B q,$$

that is $P\otimes_B Q$ is a right H-comodule. Let $h\in H$, then:

$$\begin{aligned}
\rho[h(p\otimes_B q)] &= \rho[(h_1p)\otimes_B (h_2q)]\\
&= [(h_1p)\otimes_B (h_2q)]_0\otimes[(h_1p)\otimes_B (h_2q)]_1\\
&= (h_1p)_0\otimes_B (h_2q)_0\otimes(h_2q)_1(h_1p)_1\\
&\overset{(4.1.9)}{=} (h_{12}p_0)\otimes_B (h_{22}q_0)\otimes(h_{23}q_1S_H^{-1}(h_{21}))(h_{13}p_1S_H^{-1}(h_{11}))\\
&= (h_2p_0)\otimes_B (h_5q_0)\otimes(h_6q_1S_H^{-1}(h_4))(h_3p_1S_H^{-1}(h_1))\\
&= (h_2p_0)\otimes_B (h_4q_0)\otimes(h_5q_1)(S_H^{-1}(h_{32})h_{31}p_1S_H^{-1}(h_1))\\
&= (h_2p_0)\otimes_B (h_4q_0)\otimes(h_5q_1)(\varepsilon(h_3)1_Hp_1S_H^{-1}(h_1))\\
&= (h_2\varepsilon(h_3)p_0)\otimes_B (h_4q_0)\otimes(h_5q_1)(1_Hp_1S_H^{-1}(h_1))\\
&= (h_{21}\varepsilon(h_{22})p_0)\otimes_B (h_3q_0)\otimes(h_4q_1)(1_Hp_1S_H^{-1}(h_1))\\
&= (h_2p_0)\otimes_B (h_3q_0)\otimes(h_4q_1)(1_Hp_1S_H^{-1}(h_1))\\
&= (h_{21}p_0)\otimes_B (h_{22}q_0)\otimes(h_3q_1)(p_1S_H^{-1}(h_1))\\
&= h_2(p_0\otimes_B q_0)\otimes h_3(q_1p_1)S_H^{-1}(h_1)\\
&= h_2(p\otimes_B q)_0\otimes h_3(p\otimes_B q)_1S_H^{-1}(h_1)
\end{aligned}$$

The relation (4.1.9) is satisfied for $P\otimes_B Q$, that is, it is a Hopf Yetter-Drinfel'd H-module. By [32, Lemma 3.4], $P\otimes_B Q$ is an object of $Dys\text{-}_S\mathcal{Q}^H$, therefore $P\otimes_B Q\in {}_A\mathcal{Q}_A^H$

ii) Same way as the proof of item *i*).

iii) • Let us consider $f\in Hom_B(P,B)$, $a,a'\in A$ $b,b'\in B$ and $p\in P$. The left B-action and the right A-action on $Hom_B(P,B)$ are respectively given by

$$(bf)(p) = bf(p) \text{ and } (fa)(p) = f(ap).$$

From this we have:

$$[(bb')f](p) = (bb')f(p) = b[b'f(p)] = b[(b'f)(p)] = [b(b'f)](p) \text{ and } (1_Bf)(p) = 1_Bf(p) = f(p),$$

$[f(aa')](p) = f((aa')p) = f(a(a'p)) = (fa)(a'p) = [(fa)a'](p)$ and $(f1_A)(p) = f(1_Ap) = f(p)$.

The compatibility between the left B-action and the right A-action is:

$$[b(fa)](p) = b(fa)(p) = (bf)(p)a = [(bf)a](p).$$

Now let $s \in S$ and $f \in Hom_B(P,B)$. For $p \in P$, we have:

$$(sf)(pb) = sf(pb) = s(f(p)b) = (sf(p))b = ((sf)(p))b,$$

sf is right B-linear that is $(sf) \in Hom_B(P,B)$ (in other words, we recall that the left S-action on $Hom_B(P,B)$ is well-defined). Let $h \in H$, the left H-action on $Hom_B(P,B)$ is given by $(h.f) = h_1[f(S_H(h_2)p)]$. This action is well-defined since, for all $b \in B$,

$$\begin{aligned}
(h.f)(pb) &= h_1[f(S_H(h_2)(pb))] \\
&= h_1[f[(S_H(h_2)_1p)(S_H(h_2)_2.b)]] \\
&= h_1[f[(S_H(h_{22})p)(S_H(h_{21}).b)]] \\
&= h_1[f(S_H(h_3)p)(S_H(h_2).b)] \\
&= [h_{11}f(S_H(h_3)p)][h_{12}(S_H(h_2).b)] \\
&= [h_1f(S_H(h_4)p)][h_2(S_H(h_3).b)] \\
&= [h_1f(S_H(h_3)p)][(h_{12}S_H(h_{22})).b] \\
&= [h_1f(S_H(h_3)p)][(\varepsilon(h_2)1_H).b] \\
&= [h_1f(S_H((\varepsilon(h_2)h_3)p)][1_H.b] \\
&= [h_1f(S_H(h_2)p)]b \\
&= [(h.f)(p)]b
\end{aligned}$$

that is, $h.f \in Hom_B(P,B)$. Now let us verify that the right H-coaction on $Hom_B(P,B)$ is well-defined:

$$\begin{aligned}
f_0(pb)_0 \otimes f_1 &\overset{(4.1.6)}{=} f((pb)_0)_0 \otimes S_H^{-1}((pb)_1)f((pb)_0)_1 \\
&= f(p_0b_0)_0 \otimes S_H^{-1}(b_1p_1)f(p_0b_0)_1 \\
&= f(p_0)_0b_{00} \otimes S_H^{-1}(p_1)S_H^{-1}(b_1)b_{01}f(p_0)_1 \\
&= f(p_0)_0b_0 \otimes S_H^{-1}(p_1)S_H^{-1}(b_{12})b_{11}f(p_0)_1 \\
&= f(p_0)_0b_0 \otimes S_H^{-1}(p_1)\varepsilon(b_1)f(p_0)_1 \\
&= f(p_0)_0b_0\varepsilon(b_1) \otimes S_H^{-1}(p_1)f(p_0)_1 \\
&= f(p_0)_0b \otimes S_H^{-1}(p_1)f(p_0)_1 \\
&= f_0(p)b \otimes f_1.
\end{aligned}$$

This means that $f_0 \in Hom_B(P,B)$. According to [32, Lemma 2.1 and Lemma 3.5 (i)], $Hom_B(P,B)$ is an object of $Dys\text{-}_S\mathcal{Q}^H$. We deduce from [32, Lemma 4.1] that $End_B(P)$ is an algebra in $Dys\text{-}_S\mathcal{Q}^H$.

- Let $f \in {}_AHom(P,A)$, $b \in B$, $a \in A$ and $p \in P$. The left B-action and the right A-action on ${}_AHom(P,A)$ are respectively given by:

$$(bf)(p) = f(pb) \text{ and } (fa)(p) = f(p)a.$$

For all $a,a' \in A$ and $b,b' \in B$ we have:

$$[(bb')f](p) = f[p(bb')] = f[(pb)b'] = (b'f)(pb) = [b(b'f)](p) \text{ and } (1_B f)(p) = f(p1_B) = f(p)$$

$$[f(aa')](p) = f(p)(aa') = (f(p)a)a' = (fa)(p)a' = [(fa)a'](p) \text{ and } (f1_A)(p) = f(p)1_A = f(p)$$

and

$$[b(fa)](p) = (fa)(pb) = f(pb)a = (bf)(p)a = [(bf)a](p),$$

that is, the left B-action and the right A-action on ${}_AHom(P,A)$ are compatible. Let $s \in S$ and $f \in {}_AHom(P,A)$;

$$(sf)(ap) = f((ap)s) = f(a(ps)) = af(ps) = a(sf)(p),$$

then sf is left A-linear. For all $a \in A,\ h \in H,\ p \in P$; we have:

$$\begin{aligned}
(h.f)(ap) &\overset{(4.1.24)}{=} h_2 f(S_H^{-1}(h_1)(ap)) \\
&= h_2 f[(S_H^{-1}(h_1)_1.a)(S_H^{-1}(h_1)_2 p)] \\
&= h_2 f[(S_H^{-1}(h_{12}).a)(S_H^{-1}(h_{11})p)] \\
&= h_3 f[(S_H^{-1}(h_2).a)(S_H^{-1}(h_1)p)] \\
&= h_3[(S_H^{-1}(h_2).a)f(S_H^{-1}(h_1)p)] \\
&= [h_{31}(S_H^{-1}(h_2).a)][h_{32} f(S_H^{-1}(h_1)p)] \\
&= [(h_{22}S_H^{-1}(h_{21})).a][h_3 f(S_H^{-1}(h_1)p)] \\
&= [(\varepsilon(h_2)1_H).a][h_3 f(S_H^{-1}(h_1)p)] \\
&= a[\varepsilon(h_2)h_3 f(S_H^{-1}(h_1)p)] \\
&= a[h_2 f(S_H^{-1}(h_1)p)] \\
&= a[(h.f)(p)]
\end{aligned}$$

Therefore we have $h.f \in {}_AHom(P,A)$. Let $a \in A$ and $p \in P$ we have:

$$\begin{aligned}
f_0(ap) \otimes f_1 &\overset{(4.1.25)}{=} f((ap)_0)_0 \otimes f((ap)_0)_1 S_H((ap)_1) \\
&= f(a_0p_0)_0 \otimes f(a_0p_0)_1 S_H(p_1a_1) \\
&= a_{00} f(p_0)_0 \otimes f(p_0)_1 a_{01} S_H(a_1) S_H(p_1) \\
&= a_0 f(p_0)_0 \otimes f(p_0)_1 a_{11} S_H(a_{12}) S_H(p_1) \\
&= a_0 f(p_0)_0 \otimes f(p_0)_1 \varepsilon(a_1) S_H(p_1) \\
&= a_0 \varepsilon(a_1) f(p_0)_0 \otimes f(p_0)_1 S_H(p_1) \\
&= af_0(p) \otimes f_1,
\end{aligned}$$

so f_0 is left A-linear, then the right H-coaction on ${}_AHom(P,A)$ is well-defined. Similarly way, according to [32, Lemma 2.2 and Lemma 3.5 (ii)], ${}_AHom(P,A)$ is an object of Dys-${}_S\mathcal{Q}^H$. From [32, Lemma 4.2], we deduce that ${}_AEnd(P)$ is an algebra in Dys-${}_S\mathcal{Q}^H$

iv) Same way as the proof of item *iii*).

$\square$

Definition 4.2.3. *A Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule which is simultaneously faithfully projective as a left A-module and faithfully projective as a right B-module will be called a Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodule faithfully projective as an $(A\text{-}B)$-bimodule. So a Hopf Yetter-Drinfeld $(S\text{-}S,H)$-bimodule faithfully projective is just a dyslectic Hopf Yetter-Drinfeld (S,H)-module faithfully projective.*

For the following definition, we refer to [14, page 317] where $S = R$ and to [59]:

Definition 4.2.4. *A* ***Morita context*** *in the braided monoidal category $(Dys\text{-}_S\mathcal{Q}^H, \otimes_S, S, \gamma)$ is a sextuple (A,B,P,Q,f,g) consisting of algebras $A,B \in Dys\text{-}_S\mathcal{Q}^H$, an $(A\text{-}B)$-bimodule $P \in Dys\text{-}_A\mathcal{Q}^H_B$, a $(B\text{-}A)$-bimodule $Q \in Dys\text{-}_B\mathcal{Q}^H_A$ and bilinear morphisms $f : P \otimes_B Q \to A$ and $g : Q \otimes_A P \to B$ such that*

$$f(p \otimes q)p' = pg(q \otimes p') \text{ and } g(q \otimes p)q' = qf(p \otimes q'), \quad \forall p,p' \in P, \quad q,q' \in Q.$$

The Morita context $\{A,B,P,Q,f,g\}$ is strict, if f and g are isomorphisms in $Dys\text{-}_S\mathcal{Q}^H$.

Theorem 4.2.5. *Let (A,B,P,Q,f,g) be a strict Morita context in $Dys\text{-}_S\mathcal{Q}^H$. Then*

i) the functors $\left\{\begin{array}{l} P \otimes_B \bullet : {}_B\mathcal{Q}^H_S \longrightarrow {}_A\mathcal{Q}^H_S \\ \\ Q \otimes_A \bullet : {}_A\mathcal{Q}^H_S \longrightarrow {}_B\mathcal{Q}^H_S \end{array}\right.$ *are inverse equivalences.*

ii) the functors $\left\{\begin{array}{l} \bullet \otimes_A P : {}_S\mathcal{Q}^H_A \longrightarrow {}_S\mathcal{Q}^H_B \\ \\ \bullet \otimes_B Q : {}_S\mathcal{Q}^H_B \longrightarrow {}_S\mathcal{Q}^H_A \end{array}\right.$ *are inverse equivalences.*

iii) If P is faithfully projective as an $(A\text{-}B)$-bimodule and Q faithfully projective as a $(B\text{-}A)$-bimodule, then the module isomorphisms

$$Hom_A(Q,A) \leftarrow P \to {}_BHom(Q,B) \quad \text{and} \quad Hom_B(P,B) \leftarrow Q \to {}_AHom(P,A)$$

are respectively isomorphisms of dyslectic Hopf Yetter-Drinfeld $(A\text{-}B,H)$-bimodules and $(B\text{-}A,H)$-bimodules.

iv) If P is faithfully projective as an $(A\text{-}B)$-bimodule and Q faithfully projective as a $(B\text{-}A)$-bimodule, then the isomorphisms

$$End_B(P) \leftarrow A \to {}_BEnd(Q) \quad \text{and} \quad End_A(Q) \leftarrow B \to {}_AEnd(P)$$

are isomorphisms of dyslectic Hopf Yetter-Drinfeld (S,H)-module algebras.

Proof.

$i)$ For all $M \in {}_A\mathcal{Q}_S^H$, $(Q \otimes_A M) \in {}_B\mathcal{Q}_S^H$, then

$$P \otimes_B (Q \otimes_A M) \cong (P \otimes_B Q) \otimes_A M \cong A \otimes_A M \cong M = Id_{{}_A\mathcal{Q}_S^H}(M)$$

and for all $M \in {}_B\mathcal{Q}_S^H$, $(P \otimes_B M) \in {}_A\mathcal{Q}_S^H$, then

$$Q \otimes_A (P \otimes_B M) \cong (Q \otimes_A P) \otimes_B M \cong B \otimes_B M \cong M = Id_{{}_B\mathcal{Q}_S^H}(M).$$

That is, $P \otimes_B \bullet$ and $Q \otimes_A \bullet$ are inverse each other.

$ii)$ We use the same way as $i)$.

$iii)$ Let us define the maps

$$\phi : P \longrightarrow Hom_A(Q, A) \quad \text{given by} \quad \phi(p)(q) = f(p \otimes q)$$

and

$$\phi' : P \longrightarrow {}_BHom(Q, B) \quad \text{given by} \quad \phi'(p)(q) = g(q \otimes p).$$

Let us also define the maps

$$\psi : Q \longrightarrow Hom_B(P, B) \quad \text{given by} \quad \psi(q)(p) = g(q \otimes p)$$

and

$$\psi' : Q \longrightarrow {}_AHom(P, A) \quad \text{given by} \quad \psi'(q)(p) = f(p \otimes q).$$

Using the dual basis of P and Q, we can prove as in [14, Proposition 1.1.5, *item 2.*] and its proof.

$iv)$ Let us consider the well-defined maps

$$\alpha : A \longrightarrow End_B(P); \;\; \alpha(a)(p) = ap, \qquad \alpha' : A \longrightarrow {}_BEnd(Q); \;\; \alpha'(a)(q) = qa$$

and

$$\beta : B \longrightarrow End_A(Q); \;\; \beta(b)(q) = bq, \qquad \beta' : B \longrightarrow {}_AEnd(P); \;\; \beta'(b)(p) = bp$$

We can prove this by using again the dual basis of P and Q as in [14, Proposition 1.1.5, *item 3.*] and its proof.

□

Let us set $A\#_S\bar{A} = A^{\#_e}$ and $\bar{A}\#_S A = {}^{\#_e}A$, where $\bar{A}$ is the H-opposite algebra of A (for more details and properties, see [32, Section 4]). Since $A^{\#_e}$ and ${}^{\#_e}A$ are algebras in $Dys\text{-}{}_S\mathcal{Q}^H$ (cf [32, Proposition 5.1]), we can consider the category ${}_{A^{\#_e}}\mathcal{Q}_S^H$ of left Hopf Yetter-Drinfeld $(A^{\#_e}, H)$-modules and the category ${}_S\mathcal{Q}_{{}^{\#_e}A}^H$ of right Hopf Yetter-Drinfeld $({}^{\#_e}A, H)$-modules. Clearly, A

is an object of these two categories: the left $A^{\#_e}$-action and the right ${}^{\#_e}A$-action on A are respectively defined by

$$(a\#\bar{b}).c = ac_0(c_1.b), \tag{4.2.3}$$

and

$$c.(\bar{a}\#b) = a_0(a_1.c)b; \quad \forall a,b,c \in A. \tag{4.2.4}$$

If $N \in Dys\text{-}_S\mathcal{Q}^H$, then the left $A^{\#_e}$-module $A \otimes_S N$ has a compatible Hopf Yetter-Drinfeld H-module action, so $N \mapsto A \otimes_S N$ defines a functor

$$F_H : Dys\text{-}_S\mathcal{Q}^H \longrightarrow {}_{A^{\#e}}\mathcal{Q}_S^H.$$

In a similar way, if $N \in Dys\text{-}_S\mathcal{Q}^H$, then the right ${}^{\#_e}A$-module $N \otimes_S A$ has a compatible Hopf Yetter-Drinfeld H-module action, so $N \mapsto N \otimes_S A$ defines a functor

$$F'_H : Dys\text{-}_S\mathcal{Q}^H \longrightarrow {}_S\mathcal{Q}^H_{{}^{\#_e}A}.$$

For the definition of the right adjoint to the functors F_H and F'_H, we refer to [49].

Proposition 4.2.6. *Let A be a dyslectic Hopf Yetter-Drinfeld (S,H)-module algebra. Then A is an Azumaya algebra in $Dys\text{-}_S\mathcal{Q}^H$ if and only if F_H and F'_H are equivalence functors.*

Proof. Let A be a dyslectic Hopf Yetter-Drinfeld (S,H)-module Azumaya algebra. Then the sextuple $(A^{\#_e} \cong End_S(A), S, A, A^*, f, g)$ is a strict Morita context in $Dys\text{-}_S\mathcal{Q}^H$. The map $f : A \otimes_S A^* \to End_S(A)$ is the canonical map, $g : A^* \otimes_{A^{\#_e}} A \to {}_{A^{\#_e}}End(A)$ is the evaluation map. It follows from Theorem 4.2.5 that F_H is an equivalence functor with inverse $A^* \otimes_{A^{\#_e}} \bullet$. We also have a strict Morita context $({}^{\#_e}A \cong_S End(A), S, A, A^*, f', g')$ in $Dys\text{-}_S\mathcal{Q}^H$. It follows from Theorem 4.2.5 that F'_H is an equivalence functor with inverse $\bullet \otimes_{{}^{\#_e}A} A^*$.

Assume that F_H and F'_H are equivalence functors. Taking $A = S$ and $B = A^{\#_e}$ in [48, Theorem 5.1], we get a strict Morita context $(A^{\#_e}, S, {}_{A^{\#_e}}Hom(A, A^{\#_e}), A, f, g)$ in $Dys\text{-}_S\mathcal{Q}^H$. It follows from Theorem 4.2.5, that A is faithfully projective as an S-module and $A^{\#_e} \cong End_S(A)$. Using the right hand version of [48, Theorem 5.1] with $A = S$ and $B = {}^{\#_e}A$, we get a strict Morita context $({}^{\#_e}A, S, A, Hom_{{}^{\#_e}A}(A, {}^{\#_e}A), f, g)$ in $Dys\text{-}_S\mathcal{Q}^H$. So A is faithfully projective as an S-module and ${}^{\#_e}A \cong {}_SEnd(A)$.

□

4.3 Generalization of the Rosenberg-Zelinsky sequence

In this section, we will present a generalization of Rosenberg-Zelinsky sequence for H-linear, H-colinear S-algebra automorphisms of dyslectic Hopf Yetter-Drinfeld (S,H)-module Azumaya algebras. The approach is based that of [39, Section 4.1] and [14, Section 13.6].

Throughout this section, let H be a Hopf algebra with bijective antipode and let S be an H-commutative dyslectic Hopf Yetter-Drinfeld H-module algebra.

We can apply the preceding theorem to obtain a generalization of the Rosenberg-Zelinsky exact sequence, as is done in [39, Section 4.1] and [14, Section 13.6].

If A is an (S,H)-algebra, let H-$\mathrm{Aut}_S(A)$ be the group of all H-linear, H-colinear S-algebra automorphisms of A. An element α of H-$\mathrm{Aut}_S(A)$ is called H-inner if $\alpha(x) = (x_1.u)x_0u^{-1}$ for all $x \in A$, for some unit u of A. An element α of H-$\mathrm{Aut}_S(A)$ is H-INNER if there is an *H-invariant, H-coinvariant* unit u of A for which the inner automorphism arising from u is equal to α ($u \in A$ is H-invariant when H acts trivially on u; i.e. $h.u = \epsilon_H(h)u$, for all $h \in H$; $u \in A$ is H-coinvariant when H coacts trivially on u; i.e. $u_0 \otimes u_1 = u \otimes 1_H$.) The subgroup of H-inner (resp., H-INNER) automorphisms of A will be denoted $H\text{-}Inn(A)$ (resp., $H\text{-}INN(A)$). $H\text{-}INN(A)$ is a subgroup of $H\text{-}Inn(A)$.

For α, $\beta \in H$-$\mathrm{Aut}_S(A)$, we denote by ${}_\alpha A_\beta$ the A-A-bimodule A which acts on left via α and on right via β. ${}_\alpha A_\beta$ will be the object of ${}_{A^{\#_e}}\mathcal{Q}^H_S$ that is, it is equal to A as a Hopf Yetter-Drinfeld (S,H)-module and has left $A^{\#_e}$-action

$$(a \otimes \bar{b}).c = (\alpha(a) \otimes \overline{\beta(b)}).c = \alpha(a)\beta(c_0)(c_1.b) \tag{4.3.1}$$

Lemma 4.3.1. *If α, β, $\gamma \in H$-$\mathrm{Aut}_S(A)$, then*

1. ${}_\alpha A_\beta \cong {}_{\gamma\alpha}A_{\gamma\beta}$ *in* ${}_{A^{\#_e}}\mathcal{Q}^H_S$;
2. ${}_1A_\alpha \otimes_A {}_1A_\beta \cong {}_1A_{\alpha\beta}$ *in* ${}_{A^{\#_e}}\mathcal{Q}^H_S$;
3. ${}_1A_\alpha \cong {}_1A_1$ *as $A^{\#_e}$-modules if and only if α is in $H\text{-}Inn(A)$; and*
4. ${}_1A_\alpha \cong {}_1A_1$ *in* ${}_{A^{\#_e}}\mathcal{Q}^H_S$ *if and only if α is in $H\text{-}INN(A)$.*

Proof. See the proof of [14, Theorem 13.6.1] or adapt the proof of [31, Lemma 5.3].

□

Definition 4.3.2. *A dyslectic Hopf Yetter-Drinfel'd (S,H)-module M is called invertible if there exists another dyslectic Hopf Yetter-Drinfel'd (S,H)-module N such that $M\tilde{\otimes}_S N \cong S$ as dyslectic Hopf Yetter-Drinfel'd (S,H)-modules. Isomorphism classes of invertible objects of $Dys\text{-}_S\mathcal{Q}^H$*

form a group under the tensor product $\tilde{\otimes}_S$ called the Picard group of dyslectic Hopf Yetter-Drinfel'd (S,H)-modules and denoted by $P\mathcal{Q}^H(S,H)$.

The isomorphism class in $Pic(S)$ (the group of isomorphism classes of invertible S-modules) represented by an invertible S-module M will be denoted by $[M]$. If it exists, the isomorphism class in $P\mathcal{Q}^H(S,H)$ represented by an object $M \in Dys\text{-}_S\mathcal{Q}^H$ will be denoted by $\{M\}$.

Theorem 4.3.3. *Let A be a dyslectic Hopf Yetter-Drinfeld (S,H)-module Azumaya algebra. Then there are exact sequences of groups*

$$1 \to H\text{-}Inn(A) \to H\text{-}\mathrm{Aut}_S(A) \xrightarrow{\Psi} Pic(S) \tag{4.3.2}$$

and

$$1 \to H\text{-}INN(A) \to H\text{-}\mathrm{Aut}_S(A) \xrightarrow{\Phi} P\mathscr{Q}^H(S,H). \tag{4.3.3}$$

The homomorphisms Ψ and Φ are respectively defined by

$$\Psi(\alpha) = [G_H({}_1A_\alpha)]$$

and

$$\Phi(\alpha) = \{G_H({}_1A_\alpha)\},$$

for every $\alpha \in H\text{-}\mathrm{Aut}_S(A)$, where G_H is the inverse of the equivalence functor $F_H : N \to A \otimes_S N$.

Proof. Ψ is simply the restriction of the map

$$\Psi : \mathrm{Aut}_S(A) \to Pic(S)$$

used in the original Rosenberg-Zelinsky exact sequence

$$1 \to Inn(A) \to \mathrm{Aut}_S(A) \xrightarrow{\Psi} Pic(S)$$

to the subgroup $H\text{-}\mathrm{Aut}_S(A)$. So exactness of the first sequence is immediate. For the other sequence, the fact that Φ is a group homomorphism follows from Lemma 4.3.1 (ii) and the fact that G_H is a functor equivalence. Furthermore, if $\alpha \in H\text{-}\mathrm{Aut}_S(A)$, we will have $G_H({}_1A_\alpha) \cong S$ if and only if ${}_1A_\alpha \cong {}_1A_1$. Therefore, $\mathrm{Ker}\Phi = H\text{-}INN(A)$. Exactness of the second sequence follows.

□

4.4 A dyslectic (S,H)-dimodule version of the Rosenberg-Zelinsky exact sequence

In this section, we give a version of the Rozenberg-Zelinsky sequence for dyslectic (S,H)-dimodules. When H is a commutative and cocommutative Hopf algebra, every Yetter-Drinfel'd H-module becomes an H-dimodule (see (2.1.9)).

Let M and N be two (S,H)-dimodules (cf. Section 2.2). Set $Hom_S(M,N)$, the set of right S-linear maps in ${}_S\mathscr{D}^H$ and ${}_SHom(M,N)$ the one of left S-linear maps in ${}_S\mathscr{D}^H$. Using [33, Lemmas: 3.2, 3.3 and 3.4]; for an (S,H)-dimodule P wich is finitely generated projective as a left S-module, the functors

$$\begin{cases} Hom_S(P,\bullet) : {}_S\mathscr{D}^H \longrightarrow {}_S\mathscr{D}^H \\ \bullet \otimes_S P : {}_S\mathscr{D}^H \longrightarrow {}_S\mathscr{D}^H \end{cases} \text{and} \begin{cases} {}_SHom(P,\bullet) : {}_S\mathscr{D}^H \longrightarrow {}_S\mathscr{D}^H \\ P \otimes_S \bullet : {}_S\mathscr{D}^H \longrightarrow {}_S\mathscr{D}^H \end{cases}$$

are two pairs of adjoint functors.

For M and N two dyslectic (S,H)-dimodules (cf. Section 2.3), if M is finitely generated projective as a right S-module, by [33, Lemma 4.5], $Hom_S(M,N)$ and ${}_SHom(M,N)$ are dyslectic (S,H)-modules. Therefore, the functors

$$\begin{cases} Hom_S(P,\bullet): Dys\text{-}_S\mathscr{D}^H \longrightarrow Dys\text{-}_S\mathscr{D}^H \\ \\ \bullet \otimes_S P : Dys\text{-}_S\mathscr{D}^H \longrightarrow Dys\text{-}_S\mathscr{D}^H \end{cases} \text{and} \begin{cases} {}_SHom(P,\bullet): Dys\text{-}_S\mathscr{D}^H \longrightarrow Dys\text{-}_S\mathscr{D}^H \\ \\ P \otimes_S \bullet : Dys\text{-}_S\mathscr{D}^H \longrightarrow Dys\text{-}_S\mathscr{D}^H \end{cases}$$

are two pairs of adjoint functors (where P is finitely generated projective as a right S-module).

Let A be a dyslectic (S,H)-dimodule Azumaya algebra (cf. Section 2.5). Using the results of Sections 4.2 and 4.3, we get similar results for dyslectic (S,H)-dimodules. The analogous equations of (4.2.3), (4.2.4) and (4.3.1) are respectively:

$$(a\#\bar{b}).c = a(b_1.c)b_0 \tag{4.4.1}$$

$$c.(a\#\bar{b}) = (c_1.a)c_0 b \tag{4.4.2}$$

$$(a\otimes\bar{b}).c = (\alpha(a)\otimes\beta(b)).c = \alpha(a)(b_1.c)\beta(b_0) \tag{4.4.3}$$

The isomorphism class in $Pic(S)$ represented by an invertible S-module M will be denoted by $[[M]]$. Let $P\mathscr{D}^H(S,H)$ be the group of isomorphism classes of dyslectic (S,H)-dimodules M for which there exists a dyslectic (S,H)-dimodule N for which $M\otimes_S N \cong S$ as dyslectic (S,H)-dimodules. If it exists, the isomorphism class in $P\mathscr{D}^H(S,H)$ represented by an object $M \in Dys\text{-}_S\mathscr{D}^H$ will be denoted by $\{\{M\}\}$.

Theorem 4.4.1. *Let A be a dyslectic (S,H)-dimodule Azumaya algebra. Then there are exact sequences of groups*

$$1 \to H\text{-}Inn(A) \to H\text{-}\mathrm{Aut}_S(A) \xrightarrow{\Psi'} Pic(S) \tag{4.4.4}$$

and

$$1 \to H\text{-}INN(A) \to H\text{-}\mathrm{Aut}_S(A) \xrightarrow{\Phi'} P\mathscr{D}^H(S,H). \tag{4.4.5}$$

The homomorphisms Ψ' and Φ' are respectively defined by

$$\Psi'(\alpha) = [[G_H({}_1A_\alpha)]]$$

and

$$\Phi'(\alpha) = \{\{G_H({}_1A_\alpha)\}\},$$

for every $\alpha \in H\text{-}\mathrm{Aut}_S(A)$, where G_H is the inverse of the equivalence $F_H : N \to A\otimes_S N$.

Proof. The proof of this theorem is similar to the proof of Theorem 4.3.3.

CONCLUSION AND PERSPECTIVES

Conclusion

Dans cette thèse nous avons introduit et défini la notion des (S,H)**-dimodules dyslectiques** et défini la catégorie $Dys\text{-}_S\mathscr{D}^H$, dont ils sont les objets. Nous avons défini les algèbres d'Azumaya de cette catégorie et établi leur groupe de Brauer $BD(S,H)$. Ce groupe généralise celui établi par Long dans [40]. Nous avons établi une relation d'anti-isomorphisme entre le groupe de Brauer $BD(S,H)$ et le groupe de Brauer $BQ(S^{op},H)$ défini dans [32]. En 1990, Tilborghs a montré qu'il y a un anti-isomorphism de groupes entre les groupes de Brauer $BD(R,H)$ et $BD(R,H^*)$ et dans cette thèse, nous avons généralisé son résultat en montrant que $BD(S,H) \to BD(S^{op},H^*)$ est un anti-isomorphisme de groupes. La partie 4 de cette thèse généralise aussi les résultats de Caenepeel, Oystaeyen et Zhang (cf, [17]) et de Caenepeel (cf, [14, Theorem 13.6.1]) sur les suites exactes de Rosenberg-Zelinsky.

Perspectives

Dans un futur immédiat, nous souhaiterons mener nos recherches dans le sens à si on peut établir un anti-isomorphisme de groupes entre le groupe de Brauer-Clifford-Long $BQ(S,H)$ des algèbres d'Azumaya des (S,H)-modules de Hopf-Yetter-Drinfeld dyslectiques défini par Guédénon et Herman dans [32] et le groupe de Brauer-Clifford-Long $BQ(S^{op},H^*)$ des algèbres d'Azumaya des (S^{op},H^*)-modules de Hopf-Yetter-Drinfeld dyslectiques où H est une algèbre de Hopf de dimension finie, H^* son dual et S^{op} l'algèbre opposée de l'algèbre de H-module de Hopf-Yetter-Drinfeld H-commutative S. Ce serait une généralisation du résultat établi par Nango (voir [46]). Nous voudrions également définir les sous-groupes de Brauer des groupes de Brauer $BQ(S,H)$ de [32] et $BD(S,H)$ de [33]. Dans la suite nous envisageons de:

- ✓ Voir si on pourra définir le **Groupe de Brauer-Clifford des algèbres dans la catégorie des $A\#_H C$-modules** : ici, A est une algèbre de H-module à gauche commutative et C une algèbre de H-comodule à gauche. On signale que l'algèbre $A\#_H C$ est une généralisation de $A\#H$.

- ✓ Voir si on pourra définir le **Groupe de Brauer-Clifford des algèbres dans la catégorie des $A\#_H C$-modules H-localement finis** où A est une algèbre de H-module à gauche commutative et C une algèbre de H-comodule à gauche.

✓ Trouver des **conditions suffisantes pour la semi-simplicité (ou complète réductibilité) de la catégorie** ${}_H\mathscr{M}_A^H$ des $(H\text{-}A)$-bimodules de Hopf. On en déduira alors la semi-simplicité de la catégorie ${}_H\mathscr{M}_H^H$. Des résultats analogues pour les catégories $\mathscr{M}_A^H$, $\mathscr{M}_H^H$, $\mathscr{Y}\mathscr{D}_H^H$ et $\mathscr{L}_H^H$ ont déjà été établis.

Plus tard nous souhaiterions orienter nos recherches à l'introduction et à l'étude des notions de coalgèbres de dimodules de Long, bialgèbre de dimodules de Long, algèbres de Hopf de dimodules de Long, de H-comodules de Hopf-Yetter-Drinfeld, de coalgèbres (bialgèbres, algèbres de Hopf) de H-modules de Hopf-Yetter-Drinfeld.

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